Arab Republic of Egypt
Ministry of Education
Book Sector

Student Book

Pure Mathematics

Differential & integral Calculus

2016 - 2017

غير مصرح بتداول هذا الكتاب خارج وزارة
التربية والتعليم

Third secondary

Student Book

Pure Mathematics

Differential & integral Calculus

Third secondary

Name : ....

Class : ....

School: ....

Authors

Mr. Kamal Younis Kabsha

Prof.Dr. Afaf Abo Elfotouh

Mr. Cerafiem Elias Skander

Introduction

بسم الله الرحمن الرحيم

We are pleased to introduce this book to show the philosophy on which the academic content has been prepared. This philosophy aims at:

Today's world is witnessing an ongoing scientific development. The future generation is required to get qualified using the developing methods of the future in order to keep matching with the gigantic progress of the different sciences. In regard to apply such a principle, the Ministry of Education is doing the best to improve the curricula through having the learner in the position of the explorer to the scientific facts. In addition, the Ministry of Education provides the proper training for the students in the scientific research fields throughout thinking in order to enable them to use their minds as tools of the scientific thinking. As we present the book of Calculus for the third secondary to be a helping for lightening the minds of the students to trace the scientific thinking and to motivate them to search and explore.

In the light of what previously mentioned, the following details have been considered:

★ The book has been divided into related and integrated units. Each unit has an introduction illustrating the learning outcomes, the unit planning guide, and the related key terms. In addition, the unit is divided into lessons where each lesson shows the objective of learning it through the title You will learn. Each lesson starts with the main idea of the lesson content. It is taken into consideration to introduce the content gradually from easy to hard. The lesson includes some activities, which relate Math to other school subjects and the practical life. These activities suit the students' different abilities, consider the individual differences throughout Discover the error to correct some common mistakes of the students, confirm the principle of working together and integrate with the topic. Furthermore, this book contains some issues related to the surrounding environment and how to deal with.

★ Each lesson contains examples starting gradually from easy to hard and containing various levels of thoughts accompanied with some exercises titled Try to solve. Each lesson ends in Exercises that contain various problems related to the concepts and skills that the students learned through the lesson.

★ Each unit ends in Unit summary containing the concepts and the instructions mentioned and General exams containing various problems related to the concepts and skills, which the student learned through the unit.

★ Each unit ends in an Accumulative test to measure some necessary skills to be gained to fulfill the learning outcome of the unit.

★ The book ends in General tests including some concepts and skills, which the student learned throughout the term.

Last but not least. We wish we had done our best to accomplish this work for the benefits of our dear youngsters and our dearest Egypt.

Contents

Unit One — Differentiation and it's Applications

1 - 1 Differentiation of trigonometric function 4

1 - 2 Implicit and parametric differentiation 9

1 - 3 Heigher deivatives of the function 14

1 - 4 Two equations of the tangent and normal to a curve 18

1 - 5 Related time rates 22

Unit summary 29

General exercises 30

Accumulative test 33

Unit Two — The Calculus of Exponential and Logarithmic Functions

2 - 1 The exponential function with the natural base and the natural logarithmic function. 36

2 - 2 Derivatives of the exponential and logarithmic functions. 41

2 - 3 Integration of the exponential and logarithmic functions. 50

Unit summary 56

General exercises 57

Accumulative test 59

Unit Three — Behavior of the Function and Curve Sketching

3 - 1 Increasing and decreasing of the functions. 62

3 - 2 Maximum and minimum values 66

3 - 3 Curve sketching 72

3 - 4 Applications on maximum and minimum values. 82

Unit summary 88

General exercises 90

Accumulative test 93

Unit Four — The Definite Integral and its Applications

4 - 1 Methods of integration. 96

4 - 2 Integration of trigonometric functions. 106

4 - 3 Definite integration. 110

4 - 4 Areas in the plane. 117

4 - 5 Volumes of solids of revolution. 125

Unit summary 132

General exercises 134

Accumulative test 136

General tests 138

Unit One

Differentiation and its Applications

Unit introduction

In your previous study to the functions, you have identified explicit functions in one variable in the form y = f (x) and the operations on these functions and their formations. You have also investigated the differentiability of the continuous function on a domain and you could find the first derivative of the algebraic functions and some trigonometric functions.

In this unit, you are going to complete the study of differentiating the trigonometric functions and identify the other functions which their variables cannot be separated where the variables relate with an implicit relation or by its definition throughout the parametric variable. This requires to learn different patterns of differentiation such as implicit differentiation and parametric differentiation which depend on the derivative of the composite function (chain rule) to differentiate the function. You will also investigate the existence of the second derivative of the function in regard to learning the higher derivatives of the function in regard to learning the higher derivatives of the function which give a hand to learn several life applications.

This unit interests in some necessary applications of the differentiation in various domains of mathematics, physics, economy and the biological sciences throughout learning the two equations of the tangent and normal on a tangent of a curve and the related time rates to help you model and solve some daily life problems.

Learning outcomes

By the end of the unit and carrying out the related activities, the student should be able to:

- Find the derivatives of the trigonometric functions sec x, csc x and cot x.
- Find the differentiation of implicit functions (explicit, implicit and parametric)
- Find the higher derivatives (second and third) of different functions and identify how to express them.
- Find the two equations of the tangent and the normal to a curve at a point lying on it as an application on differentiation.
- Find the related time rates including the physical application
- Model and solve life and economic problems.

Key terms

- *Differentiation*
- *First Derivative*
- *Trigonometric Function*
- *Explicit Function*
- *Implicit function*
- *Parameter*
- *Implicit Defferentiation*
- *Parametric Defferentiation*
- *Higher Derivatives*
- *Slope of the Tangent*
- *Equation of the Tangent*
- *Equation of the Normal*
- *Rate*
- *Related Rates*

Materials

- *Graphical calculator*
- *Geogebra and graph computer*

Unit lessons

Lesson (1 - 1): Differentiation of trigonometric function

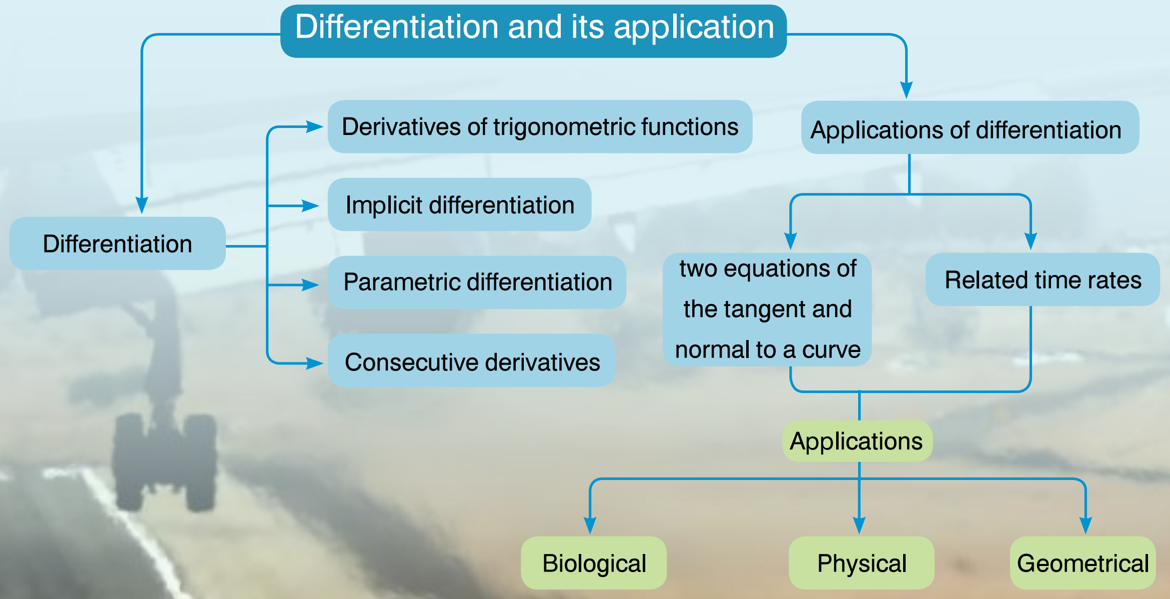
Lesson (1 - 2): Implicit and parametric differentiation

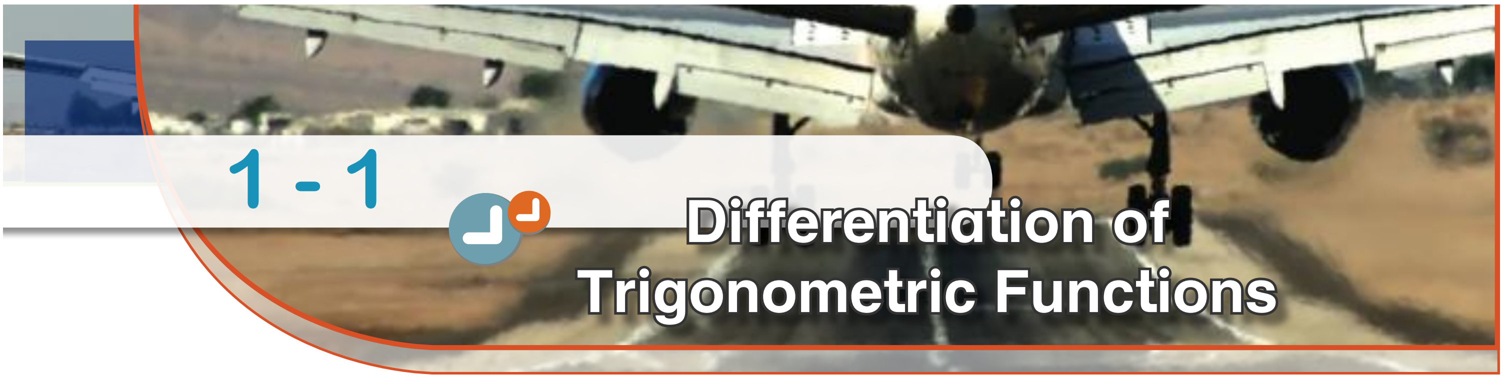
Lesson (1 - 3): Heigher deivatives of the function

Lesson (1 - 4): Two equations of the tangent and normal to a curve

Lesson (1 - 5): Related time rates

Unit planning guide





1 - 1 Differentiation of Trigonometric Functions

You will learn

Finding a derivative:

- $f(x) = \cot x$
- $f(x) = \sec x$
- $f(x) = \csc x$

Key terms

- Trigonometric function
- Drivative
- cot(x)
- Sec(x)
- Csc(x)

Materials

- scientific calculator
- computer graphic

f(x)	f'(x)
a (a∈ R)	Zero
x^n (n ∈ R)	$n x^{n-1}$
sin x (x^{rad})	cos x
cos x (x^{rad})	- sin x
tan x (x^{rad})	$\sec^2 x$

Introduction:

You have previously learned the differentiation of some trigonometric functions. In this lesson, we are going to recall you with some basic concepts about differentiation to identify the derivatives of other trigonometric functions.

Think and discuss

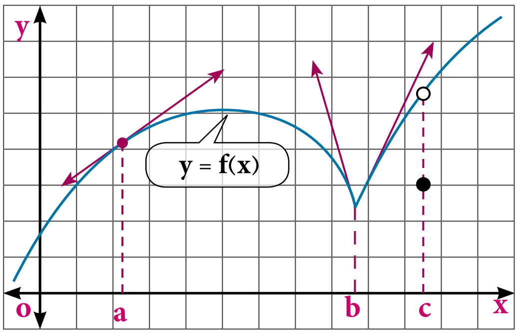
You know the rate of change of the function f at point (a , f(a)) = $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ in a condition the limit is existed. It is also the derivative of the function f at the same point, it is denoted by the symbol

f' (a) and it is the slope of the tangent to the curve of f at point (a, f(a)) = f' (a)

From your graphical reading of the opposite figure, discuss the differentiability of the function f at:

x = c , x = a , x = b ,

x ∈]a , b[and x∈ [a , b]



Note that :

The slope of the tangent is known for all the points of the curve except for x = b and x = c, then

1- The function f is discontinuous at x = c, so it is not differentiable at c.

2- The function f is differentiable at x = a since f' (a) is existed

3- The function f is not differentiable at x = b since the right derivative $f'(b^+) \neq$ the left derivative $f'(b^-)$

4- The function f is differentiable on the interval]a , b[if a derivative has been found at each point belongs to this interval for the function.

5- The function f defined on the closed interval [a , b] is differentiable if and only if each of $f'(a^+)$ and $f'(b^-)$ are existed and the function is differentiable on the interval]a , b[

Remember : the differentiation rules

1- $\frac{d}{dx}$ [a f(x)] = a f' (x) , a ∈ R

2- $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

3- $\frac{d}{dx}[f(x) \times g(x)] = f(x) \bullet g'(x) + g(x) \bullet f'(x)$

4- $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \bullet f'(x) - f(x) \bullet g'(x)}{[g(x)]^2}$, $g(x) \neq 0$

If y = f(z) is differentiable with respect to z, z = g (x) is differentiable with respect to x ,

then y = f [g (x)] is differentiable with respect to x

i.e. : $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$ **[Chain rule]**

i.e. : $\frac{d}{dx} f[g(x)] = f'[g(x)] \bullet g'(x)$

if f is a differentiable function with respect to x and n is a real number ,

then: $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \times f'(x)$

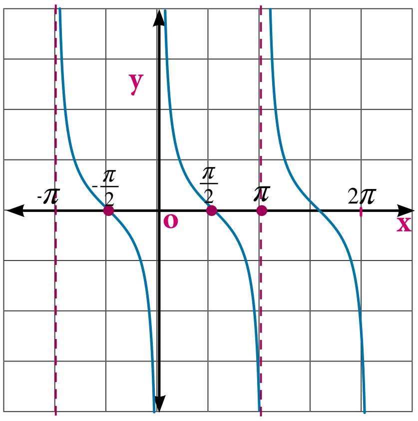
i.e. if y = f(x), **then** $\frac{d}{dx}(y)^n = n\, y^{n-1} \frac{dy}{dx}$

Learn

1 - Derivative of cotangent function

If y = cot x where $x \in R$, $x \neq n\pi$, $n \in z$

then: $\frac{d}{dx}(\cot x) = -\csc^2 x$



y = cot x

Notice that :

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right]$$

$$= \frac{\sin x \times (-\sin x) - \cos x \times \cos x}{(\sin x)^2} = -\frac{1}{(\sin x)^2} = -\csc^2 x$$

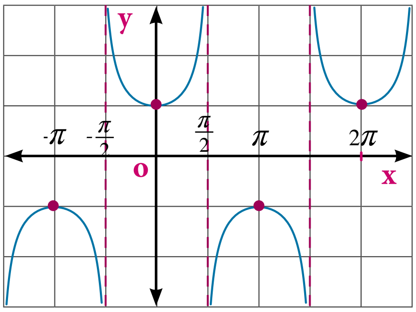
Derivative of the secant function

If y = sec x where :

$x \in R$, $x \neq \frac{(2n+1)\pi}{2}$, $n \in Z$

then:

$\frac{d}{dx}(\sec x) = \sec x \tan x$ **(check)**



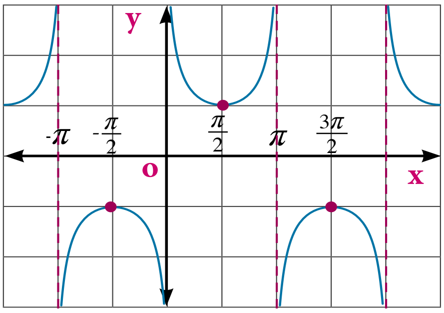
y = sec x

3- Derivative of cosecant function:

If y = csc x where

$x \in R$, $x \neq n\pi$, $n \in Z$

Then : $\frac{d}{dx}(\csc x) = -\csc x \cot x$ **check**



y = csc x

Example

1 Find $\frac{dy}{dx}$ for each of the following:

a $y = 3x^5 + 4\cot x$

b $y = 3\sec x - 5\tan x$

c $y = x^3 \csc x$

d $y = \frac{1 - \cot x}{1 + \cot x}$

Solution

a $\frac{dy}{dx} = 3 \times 5x^4 + 4(-\csc^2 x) = 15x^4 - 4\csc^2 x$

b $\frac{dy}{dx} = 3(\sec x \tan x) - 5\sec^2 x = \sec x[3\tan x - 5\sec x]$

c $\frac{dy}{dx} = 3x^2 \csc x + x^3(-\csc x \cot x) = x^2 \csc x[3 - x\cot x]$

d $\frac{dy}{dx} = \frac{(1 + \cot x)(\csc^2 x) - (1 - \cot x)(-\csc^2 x)}{(1 + \cot x)^2}$

$= \frac{\csc^2 x[1 + \cot x + 1 - \cot x]}{(1 + \cot x)^2} = \frac{2\csc^2 x}{(1 + \cot x)^2}$

Try to solve

1 Find $\frac{dy}{dx}$, if y equals:

a $2\sin x - 3\cot x$

b $\cos x + 4\sec x$

c $\sec x \tan x$

d $\csc x \cot x$

e $\frac{\sec x}{1 + \sec x}$

f $\frac{1 - \csc x}{1 + \csc x}$

Example

2 Find $\frac{dy}{dx}$ for each of the following:

a $y = \sec(5x + 2)$

b $y = \cot(\cos 3x)$

c $y = \csc^2(1 + 3x^2)$

d $y = (3 - 2\cot x)^3$

Solution

a $\because y = \sec(5x + 2)$ Putting $z = 5x + 2$ $\therefore \frac{dz}{dx} = 5$

and $y = \sec z$ where $\frac{dy}{dz} = \sec z \tan z$

$\because \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$ [chain rule]

$\therefore \frac{dy}{dx} = \sec z \tan z \times 5 = 5 \sec(5x + 2) \tan(5x + 2)$

Another Solution

$\because \frac{d}{dx} f[g(x)] = f'[g(x)] \times g'(x)$

$\therefore \frac{dy}{dx} = \sec(5x + 2) \tan(5x + 2) \frac{d}{dx}(5x + 2)$

$= 5 \sec(5x + 2) \tan(5x + 2)$

b $y = \cot(\cos 3x)$

$\therefore \frac{dy}{dx} = -\csc^2(\cos 3x) \frac{d}{dx}(\cos 3x)$

$= -\csc^2(\cos 3x) \times [-\sin 3x \times 3] = 3 \sin 3x \csc^2(\cos 3x)$

c $y = \csc^2(1 + 3x^2)$

$\therefore \frac{dy}{dx} = 2 \csc(1 + 3x^2) \times -\csc(1 + 3x^2) \cot(1 + 3x^2) \times 6x$

$= -12x \cot(1 + 3x^2) \csc^2(1 + 3x^2)$

d $y = (3 - 2\cot x)^3$

$\frac{dy}{dx} = 3(3 - 2\cot x)^2 \times -2(-\csc^2 x) = 6\csc^2 x (3 - 2\cot x)^2$

Try to solve

2) Find $\frac{dy}{dx}$ if y equals:

a $\cot(x^2 + 3)$

b $\sec\sqrt{x - 2}$

c $\sin(\csc 3x^2)$

d $3 \sec 2x + 2 \csc 3x$

e $-\csc^3(2x + \pi)$

f $x^2 \sec \frac{1}{x}$

Exercises 1 - 1

If f, g and h are differentiable functions with respect to x, complete the following using the values given in the opposite table:

1. $h(x) = 3f(x) - 2g(x)$ then $h'(1) =$ ....
2. $h(x) = f(x)[5 + g(x)]$ then $h'(2) =$ ....
3. $h(x) = f(x) \div [g(x) + 2]$ then $h'(1) =$ ....
4. $h(x) = f[g(x)]$ then $h'(1) =$ ....
5. $h(x) = g[3x - f(x)]$ then $h'(2) =$ ....
6. $h(x) = [x^3 + g(x)]^{-2}$ then $h'(1) =$ ....

x	1	2
f(x)	- 1	4
g(x)	2	1
f'(x)	1	5
g'(x)	2	- 3

Find $\frac{dy}{dx}$ for each of the following:

7. $y = x^3 - 2\sec x$
8. $y = \csc(2 - 3x)$
9. $y = \cot(\pi - \frac{1}{x})$
10. $y = \tan(\cot x)$
11. $y = (1 + \cot x)^2$
12. $y = \csc(\pi - 2x)$
13. $y = \cos 2x - 5\cot 3x$
14. $y = \tan 3x - \csc 2x$
15. $y = \sin^2 3x + \sec^2 x$.
16. $y = \sec x \tan 2x$
17. $y = \cot^3 \sqrt{x}$
18. $y = \csc^2(1 + x^2)$
19. $y = 3\sec^2(2x + \pi)$
20. $y = \sqrt{1 + \csc x}$
21. $y = x^2 \cot 3x$
22. $y = (\csc x + \cot x)^{-1}$
23. $y = \frac{\cot 3x}{2x + 3}$
24. $y = \frac{1 - \sec x}{1 + \sec x}$

Answer the following:

25. If $y = \cot \frac{\pi}{6} z$, $z = 3\sqrt{x}$ find $\frac{dy}{dx}$ at $x = 1$
26. If $y = \sqrt{5 - 2z}$, $z = \sec 2x$ prove that $\sqrt{3}\frac{dy}{dx} + 12 = 0$ at $x = \frac{\pi}{6}$
27. Find the slope of the tangent to the curve of the function f where y = f(x) for each of the following:
    - **a** $y = 2\cot x + \sqrt{2}\sec x$ at $x = \frac{\pi}{4}$
    - **b** $y = 3\tan x - \csc^2 x$ at $x = \frac{3\pi}{4}$

Implicit and Parametric Differentiation

1 - 2

Implicit Defferentiation

You have previously found a defined function in the form y = f(x); it is an explicit function of the independent variable x where it determines the value of y directly whenever the value of x is known such as :

$y = 4x^3 - 5x + 2$, $y = \sin(2x + 3)$, $y = \frac{x+1}{x-1}$, ...

and $y' = 12x^2 - 5$, $y' = 2\cos(2x+3)$, y' = ...

But if y is related to the variable x with an equation containing x and y together such as:

$xy + y - 4 = 0$ (1) , $x^2 + y^2 - 9 = 0$ (2)

then each equation defining an implicit relation between x , y expresses the relation between the two coordinates of point (x, y) lying on its graphical curve .

Notice that:

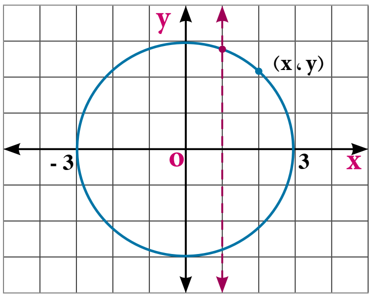
1- the equation $xy + y - 4 = 0$ can be written in the form:

$y(x + 1) = 4$ $\therefore y = \frac{4}{x+1}$ where $x \neq -1$

in this case, the implicit relation is defined as an explicit function.

2- The set of points (x , y) which satisfies the equation $x^2 + y^2 = 9$ draws a circle whose center is the origin point and its radius length is 3 units. From testing the vertical line, we notice the relation $x^2 + y^2 = 9$ doesnot represent a function but $y^2 = 9 - x^2$

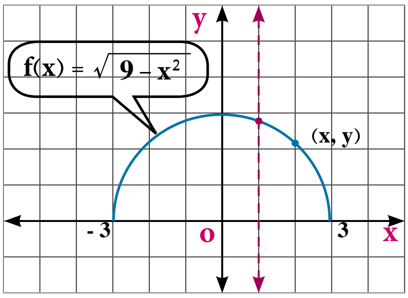
$\therefore y = \pm\sqrt{9 - x^2}$



The implicit relation $x^2 + y^2 = 9$ can be defined as two explicit functions

the first $y = \sqrt{9 - x^2}$

its domain is [-3 , 3] range is [0 , 3] and it is differentiable for each $x \in]-3 , 3[$



You will learn

- implicit differentiation
- Parametric differentiation

Key terms

- Relation
- Explicit function
- Implicit function
- Parameter

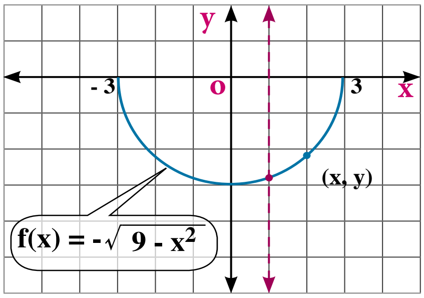
Materials

- Scientific calculator

The second: $y = -\sqrt{9 - x^2}$

its domain [-3, 3] range [-3, 0]

and it is differentiable for each $x \in]-3, 3[$



in many equations in the form f(x , y) = 0 it is difficult to express y in terms of x directly since the variable y doesnot represent an explicit function with respect to x. This function is called the implicit function.

The process of differentiating the implicit function (implicit differentiation) requires to differentiate both sides of the equation with respect to one of the two variables x or y according to the chain rule to get $\frac{dy}{dx}$ or $\frac{dx}{dy}$ respectively.

Example

1 Find $\frac{dy}{dx}$ if:

a $x^3 + y^2 - 7x + 5y = 8$

b $3xy + y^2 = x^2 - 7$

Solution

a Notice that the equation doesnot give y explicity in terms of x. To find $\frac{dy}{dx}$, we differentiate the two sides of the equation with respect to x taking into consideration that y is a function to the variable x and it is differentiable then :

$$3x^2 + 2y\frac{dy}{dx} - 7 + 5\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y + 5) = 7 - 3x^2 \qquad \therefore \frac{dy}{dx} = \frac{7 - 3x^2}{2y + 5}$$

Remember

If y is a function in x and is differentiable ،then:

$$\frac{d}{dx}(y)^n = n\,y^{n-1}\frac{dy}{dx}$$

b $\because 3xy + y^2 = x^2 - 7$ by differentiating the two sides of the equation with respect to x.

$$\therefore \frac{d}{dx}(3xy) + 2y\frac{dy}{dx} = 2x$$

$$3x\frac{dy}{dx} + y \times 3 + 2y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}[3x + 2y] = 2x - 3y \qquad \therefore \frac{dy}{dx} = \frac{2x - 3y}{3x + 2y}$$

Try to solve

1 Find $\frac{dy}{dx}$ if:

a $x^3 - 5xy + y^3 = 4x$

b $x^2y + y^2x = 25$

Example

2 Find $\frac{dy}{dx}$ if:

a $\sin 2y = y\cos 3x$

b $\tan 2x + \cot y = xy$

Solution

a By differentiating both sides of the equation with respect to x

$\therefore \frac{d}{dx}(\sin 2y) = \frac{d}{dx}(y \cos 3x)$

$\cos 2y \times 2\frac{dy}{dx} = y[-\sin 3x \times 3] + \cos 3x\,[\frac{dy}{dx}]$

$\frac{dy}{dx}[2\cos 2y - \cos 3x] = -3y\sin 3x \qquad \therefore \frac{dy}{dx} = \frac{3y\sin 3x}{\cos 3x - 2\cos 2y}$

b By differentiating both sides of the equation with respect to x

$\frac{d}{dx}(\tan 2x) + \frac{d}{dx}(\cot y) = \frac{d}{dx}(xy)$

$2\sec^2 2x - \csc^2 y\frac{dy}{dx} = x\frac{dy}{dx} + y$

$\frac{dy}{dx}[x + \csc^2 y] = 2\sec^2 2x - y \qquad \therefore \frac{dy}{dx} = \frac{2\sec^2 2x - y}{x + \csc^2 y}$

Try to solve

2 Find $\frac{dy}{dx}$ if:

a $x\cos y + y\cos x = 1$ **b** $3y = \sin x \cos 2y$

Notice: The final formula of the derivative $\frac{dy}{dx}$ in the implicit differentiation contains both x and y. This makes the process of calculating it over difficult at one of the values of x because we need first to know the corresponding value of y which is so difficult to be determined from the implicit relation.

Parametric Defferentiation

If the x-coordinate and y- coordinate of point (x , y) can be expressed as a function in a third variable t (it is called the parameter) by the two equations:

$x = f(t)$ and $y = g(t)$ where f and g have the same domain.

the two equation together represent an equation to one curve expressed in the parametric form

Learn

For the curve given in the parametric form $x = f(t)$, $y = g(t)$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ where f and g are two differentiable functions with respect to t

Example

3 Find $\frac{dy}{dx}$ for the following curves at the given values:

a $x = 5t + 3$, $y = 16t^2 + 9$, $t = 5$ **b** $x = 3\cos 2\theta$, $y = 4\sin 3\theta$, $\theta = \frac{\pi}{4}$

Solution

a $x = 5t + 3 \qquad \frac{dx}{dt} = 5, \qquad y = 16t^2 + 9 \qquad \frac{dy}{dt} = 32t$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{32t}{5} \qquad$ and $\left[\frac{dy}{dx}\right]_{t=5} = 32$

b $x = 3\cos 2\theta \qquad \frac{dx}{d\theta} = 3 \times -\sin 2\theta \times 2 = -6\sin 2\theta$

$y = 4\sin 3\theta \qquad \frac{dy}{d\theta} = 4 \times \cos 3\theta \times 3 = 12\cos 3\theta$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{12\cos 3\theta}{-6\sin 2\theta} = \frac{-2\cos 3\theta}{\sin 2\theta}$

at $\theta = \frac{\pi}{4}$ then $\frac{dy}{dx} = \frac{-2\cos\frac{3\pi}{4}}{\sin\frac{\pi}{2}} = -2 \times \frac{-1}{\sqrt{2}} = \sqrt{2}$

Try to solve

3) Find $\frac{dy}{dx}$ for the following curves at the given values

a $x = (t + 7)(t - 2) \quad , \quad y = (t^2 + 1)(t - 2) \quad , \quad t = 1$.

b $x = \sec^2\theta - 1 , \quad y = \tan\theta , \quad \theta = \frac{-3\pi}{4}$

c $x = \sqrt{3t - 2} \quad , \quad y = \sqrt{4t + 1} \quad , \quad t = 2$

Critical thinking: find the value of the parameter z at which the curve $x = 2z^3 - 5z^2 - 4z + 12 \quad , \quad y = 2z^2 + z - 4$ has a horizontal tangent and another vertical tangent.

Example

4) Find the derivative of $(4x^3 - 9x^2 + 5)$ with respect to $(3x^2 + 7)$

Solution

Putting $y = 4x^3 - 9x^2 + 5 , \quad z = 3x^2 + 7$ then $y = f(x), \qquad z = g(x)$

the two function f, g are differentiable with respect to x considering x is parameter for each of the two variables y and z

$\therefore$ from the parametric differentiation , we find that:

$\frac{dy}{dz} = \frac{y'}{z'} = \frac{12x^2 - 18x}{6x} = 2x - 3 \qquad$ **i.e.** $\frac{d}{d(3x^2 + 7)}[4x^3 - 9x^2 + 5] = 2x - 3$

Try to solve

4) Use the parametric differentiation to find:

a derivative $x^2 + 1$ with respect to $\sqrt{x^2 - 1}$

b derivative $\sqrt{8 + x^2}$ with respect to $\frac{x}{x + 1}$ at $x = 1$

c derivative $x - \sin x$ with respect to $1 - \cos x$ at $x = \frac{\pi}{3}$

Exercises 1 - 2

First : choose the correct answer:

1. If $x^2 + y^2 = 1$, then $\frac{dy}{dx}$ equals:

   a) x b) $\frac{1}{y}$ c) $-\frac{x}{y}$ d) $-\frac{y}{x}$

2. If $x^2 + y^2 = 2xy$, then $\frac{dy}{dx}$ equals:

   a) -1 b) 0 c) 1 d) 2

3. If $y^2 - 2\sqrt{x} = 0$, then $\frac{dy}{dx}$ equals:

   a) $\frac{2y}{\sqrt{x}}$ b) $\sqrt{x}$ c) $\frac{x}{y^2}$ d) $\frac{1}{y^3}$

4. If $x = 2t^2 + 3$, $y = \sqrt{t^3}$, $n = 1$, then $\frac{dy}{dx}$ equals:

   a) $\frac{3}{8}$ b) $\frac{3}{4}$ c) 2 d) 6

5. the slope of the tangent to the curve $x y^2 = 3$ at point $(3, 1)$ equals:

   a) -3 b) $-\frac{1}{6}$ c) $\frac{1}{3}$ d) $\frac{3}{2}$

Second: find $\frac{dy}{dx}$ for each of the following:

6. $x^2 - 4y^2 + 7 = 0$
7. $x^4 + 3y^4 - 2 = 0$
8. $x^2 - 2xy = 5 - y^2$
9. $x^3 + 6xy = 4y + 3$
10. $\frac{x}{y} + \frac{y}{x} = 1$
11. $xy + \sin y = 5$
12. $x \sin y + y \cos x = 0$
13. $x \csc y = y \cot x$
14. $x^2 \sin y - y^2 \sin x = 9$
15. $\sin 2x \cos 2y = \frac{3}{4}$

Third: find $\frac{dy}{dx}$ for the following curves at the given values:

16. $x = 13 - 2t$, $y = 4t^2 - \sqrt{t}$, $t = 4$
17. $x = \sin 2\pi\theta$, $y = \cos 2\pi\theta$, $\theta = \frac{1}{6}$
18. $x = 5 + \sec^2 3\theta$, $y = 1 - \tan 3\theta$, $\theta = \frac{\pi}{4}$
19. Find the slope of the tangent to the curve $\cos\sqrt{\pi y} = 3x + 1$ at point $(-\frac{1}{3}, \frac{\pi}{4})$
20. Find the derivative of $\frac{x+1}{x-1}$ with respect to $\sqrt{2x+1}$ at $x = 4$
21. Find the value of parameter t at which the curve $x = 2t^3 - 5t^2 + 4t - 9$, $y = 2t^2 + n - 5$ has:

    a) a vertical tangent. b) a horizontal tangent.

1 - 3 Higher Derivatives of a Function

You will learn

- *Finding the higher order derivatives of a function*

Key terms

- *Order*
- *Derivative*

Materials

- *Scientific calculator*

Think and discuss

if $y = f(x)$ where $y = x^4 + 5x^3 - 2x + 3$ find the derivative of function f, Can you repeat the process of differentiation with respect to x? why? Does the process of differentiation end up? Explain.

Learn

Higher - Order Derivative

- **If** $y = f(x)$ where f is a differentiable function with respect to x, then its first derivative is $y' = \frac{dy}{dx} = f'(x)$ and it represents a new function.
- **If** the first derivative is differentiable with respect to x, then its derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the second derivative of the function f and it represents another function. It is denoted by the symbol $y'' = \frac{d^2y}{dx^2} = f''(x)$
- By repeating the process of differentiation , we get the third derivative of the function f and it is denoted by the symbol $\frac{d^3y}{dx^3}$, and so on.

The derivatives of a function starting from the second derivative are called the higher derivatives and the derivative is written from the order n as follows:

$$y^{(n)} = \frac{d^ny}{dx^n} = f^{(n)}(x) \qquad \text{where } n \text{ is a positive integer number}$$

Notice that :

1- $\frac{d^2y}{dx^2}$ is read d two y by dx squared

2- There is a difference between $\frac{d^2y}{dx^2}$ and $\left(\frac{dy}{dx}\right)^2$, the first refers to the second derivative of the function whereas the second refers to the square of the first derivative.

Example

1 Find the second derivative for each of the following:

a $y = 2x^4 + 3x - 5$ **b** $y = \frac{x + 1}{x - 1}$

c $y = \sin(3x - 2)$ **d** $y = \sqrt{3x - 2}$

Solution

a $\because y = 2x^4 + 3x - 5,\ x \in R$ $\therefore \frac{dy}{dx} = 8x^3 + 3$, $\frac{d^2y}{dx^2} = 24x^2$

b $\because y = \frac{x+1}{x-1},\ x \neq 1$ $\therefore \frac{dy}{dx} = \frac{x - 1 - (x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2},\ x \neq 1$

$\frac{d^2y}{dx^2} = \frac{d}{dx}[-2(x-1)^{-2}] = \frac{4}{(x-1)^3},\ x \neq 1$

c $\because y = \sin(3x - 2),\ x \in R$ $\therefore \frac{dy}{dx} = 3\cos(3x - 2)$, $\frac{d^2y}{dx^2} = -9\sin(3x - 2)$

d $\because y = \sqrt{3x - 2},\ x \geqslant \frac{2}{3}$ $\therefore \frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}},\ x > \frac{2}{3}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{3}{2}(3x-2)^{-\frac{1}{2}}\right] = -\frac{9}{4\sqrt{(3x-2)^3}},\ x > \frac{2}{3}$

Try to solve

1) Find the third derivative for each of the following:

a $y = x^4 - 2x^2 + 5$ **b** $z = (2n - 1)^4$

c $f(x) = \cos(2x + \pi)$ **d** $f(x) = \frac{x}{x-1}$

Critical thinking: if y = sin a x, investigate the consecutive differentiation pattern and find $y^{(25)}$

Example

2) If $y^2 + 2xy = 8$, prove that : $(x + y)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = \text{zero}$

Solution

a $\because y^2 + 2xy = 8$, **by differentiating both sides with respect to x**

$\therefore 2y\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$ **by dividing by 2**

$(x + y)\frac{dy}{dx} + y = 0$ **by differentiating both sides with respect to x**

$\therefore (x + y)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(1 + \frac{dy}{dx}\right) + \frac{dy}{dx} = 0$

and $(x + y)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = \text{zero}$

Try to solve

2) **a** If $x^2 + y^2 = 9$, **prove that:** $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$

b If $y = \tan x$, **prove that:** $\frac{d^2y}{dx^2} = 2y(1 + y^2)$

Parametric equations:

Example

(3) If $x = 2n^3 - 5$, $y = 6n^2 + 1$ find $\frac{d^2y}{dx^2}$ at $n = 1$

Solution

By differentiating each of x and y with respect to parameter n

$\therefore \frac{dx}{dn} = 6n^2$, $\frac{dy}{dn} = 12n$

$\because \frac{dy}{dx} = \frac{dy}{dn} \times \frac{dn}{dx}$

$\therefore \frac{dy}{dx} = \frac{12n}{6n^2} = 2n^{-1}$, $n \neq 0$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}[2n^{-1}] = -2n^{-2} \times \frac{dn}{dx}$

$= \frac{-2}{n^2} \times \frac{1}{6n^2} = -\frac{1}{3n^4}$, $n \neq 0$

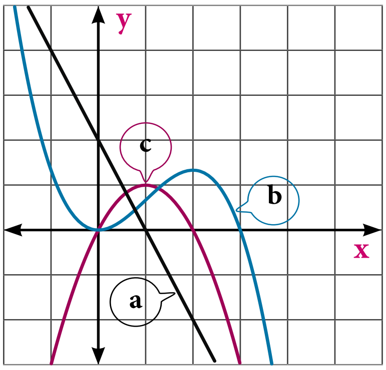
at $n = 1$ $\therefore \frac{d^2y}{dx^2} = -\frac{1}{3}$

Try to solve

(3) If $x = z^2 - 2z$, $y = z^2$

Find $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ at $z = 2$

Critical thinking: the opposite figure shows a graphical representation to the curves of the functions f(x), f' (x) , f'' (x) where f(x) is polynomial . Determine the curve of each function.



Activity

Use the geogebra program or any other program to graph the following functions and their first and second derivatives, then record your observations.

(a) $f(x) = x^3 - 4x^2 + 12$ (b) $g(x) = \frac{1}{4}x^2 + 4$

Do your observations match with your decision in critical thinking?

Exercises 1 - 3

1 If $y = ax^3 + b x^2 + c x + d$ and the correlations of x , y , y' , y'' are shown in the next table : find the real values of a , b , c , d and complete the table.

x	y	y'	y''
1	8		28
2		75	34

Find the third derivative for each of the following:

2 $y = x^5 - 4x^3 + 3$

3 $y = \frac{2x}{x+1}$

4 $y = \sin(2x - 7)$

5 $y = \cos(\pi - 3x)$

6 $y = \sin x \cos x$

7 $y = \sqrt{2x - 5}$

Answer the following:

8 If $3x^2 + 5 = 2xy$, prove that: $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 3$

9 If $x^2 + y^2 = 4$, prove that: $y^3 \frac{d^2y}{dx^2} - 4 = 0$

10 If $y = 3\cos(2x + 1)$, prove that: $\frac{d^2y}{dx^2} + 4y = 0$

11 If $xy = \sin x \cos x$, prove that: $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 4xy = 0$

12 If $y = x \sin x$, prove that: $x \frac{d^3y}{dx^3} + x \frac{dy}{dx} + 2y = 0$

13 If $y = \sec x$, prove that: $y \frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 = y^2(3y^2 - 2)$

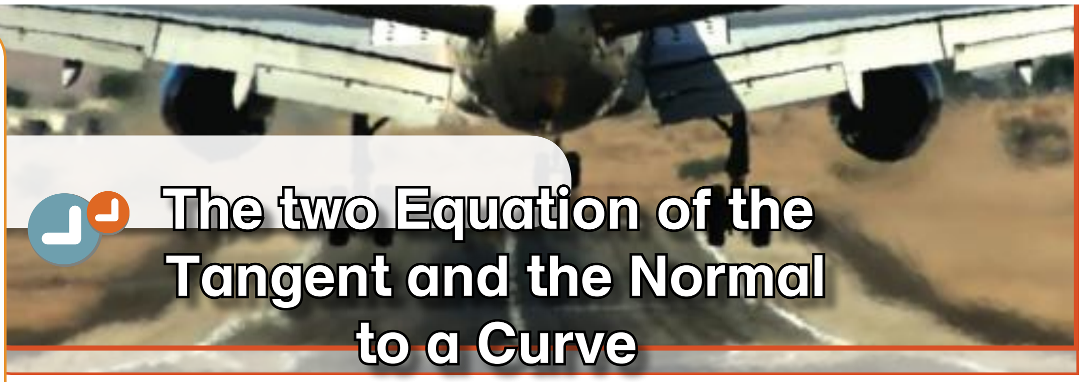
14 If $\frac{dy}{dx} = 2x - 3$, $\frac{dz}{dx} = x^2 - 1$, find: $\frac{d^2y}{dz^2}$ at $x = 2$

15 If $x = 3n^2 - 1$, $y = n^3 + 2$, find: $\frac{d^2y}{dx^2}$ at $n = 4$

16 If $x = \frac{z+1}{z-1}$, $y = \frac{z-1}{z+1}$, find: $\frac{dy}{dx}$ at $z = 2$

17 If $x = \sec z$, $\sqrt{y} = \tan z$, prove that: $\frac{d^2y}{dx^2} = 2$

1 - 4



The two Equation of the Tangent and the Normal to a Curve

You will learn

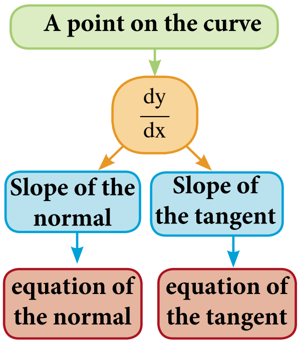
- Finding the equation of the tangent at a point lying on the curve
- Finding the equation of the normal to a curve at a point lying on the curve

Key terms

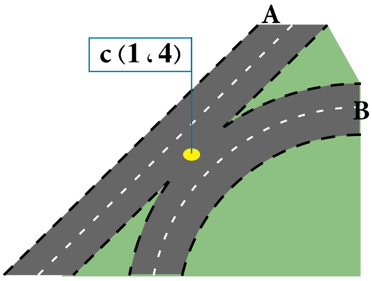
- Slope of the tangent
- Slope of the normal

Materials

- Scientific calculator
- Computer graphics



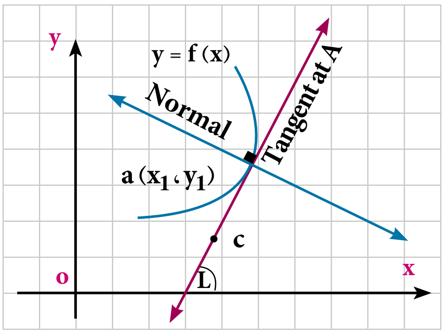
Think and discuss



The opposite figure shows two roads A and B, one road is straight and the other is curved and they are converged at position C . If position C is represented by point C (1, 4) in a perpendicular coordinate plane and the equation of the road B is : $y = 2x^2 - 3x + 5$, can you find the equation of road A?

Does the road A pass through the point (7 , 10)? Explain.

Learn



If point A(x_1 , y_1) lies on the curve of function f where y = f(x) , and m is the slope of the tangent to the curve at this point, then :

1- The equation of the tangent to the curve at point (x_1 , y_1) is : $y - y_1 = m (x - x_1)$

2- The equation of the normal to the curve at point (x_1, y_1) is : $y - y_1 = \frac{-1}{m} (x - x_1)$

Example

Trigonometric functions

(1) Find the two equations of the tangent and the normal to the curve $y = 2x - \cot x$ at the point which lies on the curve and its x-coordinate equals $\frac{\pi}{4}$

Solution

To find a point lying on the curve when $x = \frac{\pi}{4}$ calculate its y-coordinate where :

$y = 2x - \cot x$ $\qquad \therefore\ y = \frac{2 \times \pi}{4} - \cot \frac{\pi}{4} = \frac{\pi}{2} - 1$

i.e. point $(\frac{\pi}{4} , \frac{\pi}{2} - 1)$ lies on the curve

The slope of the tangent of the curve at any point

$= \frac{dy}{dx} = 2 - (-\csc^2 x) = 2 + \csc^2 x$

$\therefore$ at point $(\frac{\pi}{4}, \frac{\pi}{2} - 1)$:

the slope of the tangent = $2 + [\csc^2 \frac{\pi}{4}] = 4$, the slope of the normal = $-\frac{1}{4}$

the equation of the tangent: $y - (\frac{\pi}{2} - 1) = 4(x - \frac{\pi}{4})$ **i.e. :** $y = 4x - \frac{\pi}{2} - 1$

the equation of the normal: $y - (\frac{\pi}{2} - 1) = \frac{-1}{4}(x - \frac{\pi}{4})$ **i.e. :** $y = -\frac{1}{4}x + \frac{9}{16}\pi - 1$

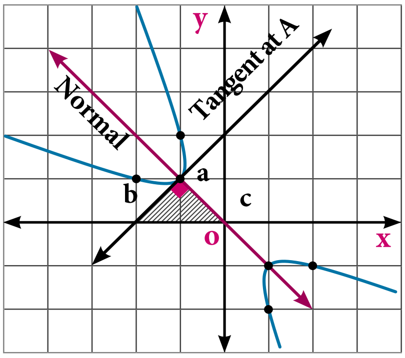
Try to solve

1. find the two equations of the tangent and the normal to the curve $y = 3 + \sec x$ at the point which lies on the curve and its x-coordinate equals $\frac{2\pi}{3}$

Example Calculating the area

2. Find the two equations of the tangent and the normal to the curve $x^2 + 3xy + y^2 + 1 = 0$ at point A (- 1, 1). If they intersect x-axis at the two points B and C, calculate the area of the triangle A B C in squared units

Solution



$\because x^2 + 3xy + y^2 + 1 = 0$

Point A (- 1, 1) satisfies the equation of the curve, it lies on the curve. By differentiating both sides of the equation of the curve with respect to x to find the slope of the tangent at any point

$\therefore 2x + 3x\frac{dy}{dx} + 3y + 2y\frac{dy}{dx} = 0$

at point A (- 1 , 1) $\therefore \frac{dy}{dx} = 1$

$\therefore$ Slope of tangent = 1 , and slope of normal = - 1

equation of tangent : $y - 1 = x + 1$ **i.e** $y = x + 2$

equation of normal : $y - 1 = -(x + 1)$ **i.e** $y = -x$

By solving the two equations of the tangent and the normal with the equation of x-axis y = 0 to find the intersection points B and C.

$\therefore$ Point B (-2, 0) , point C (0 , 0) and B C = 0 - (- 2) = 2

Area of triangle A B C = $\frac{1}{2} \times 2 \times 1 = 1$ squared units

Try to solve

2. Find the area of the triangle bounded by x-axis , tangent and normal to the curve $3x^2 + y^2 = 12$ at point (- 1 , 3)

Example **Parametric differentiation**

3 The two parametric equations of a curve are $x = t^2 + 2$, $y = t^3 + 1$

Find the two equations of the tangent and the normal to the curve when t = 1

Solution

The slope of the tangent at any points $= \frac{dy}{dx}$ where :

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3t^2}{2t} = \frac{3}{2} t$$

when t = 1, the slope of the tangent $= \frac{3}{2}$ and the slope of the normal $= -\frac{2}{3}$

$\therefore\ x = (1)^2 + 2 = 3$, $y = (1)^3 + 1 = 2$

i.e point (3 , 2) lies on the curve and then :

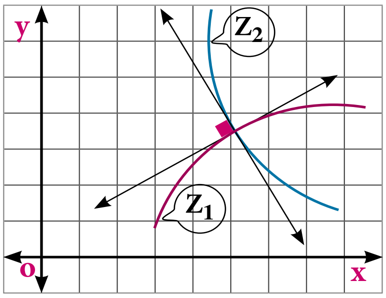
the equation of the tangent : $(y - 2) = \frac{3}{2}(x - 3)$ **i.e** $3x - 2y - 5 = 0$

the equation of normal : $(y - 2) = -\frac{2}{3}(x - 3)$ **i.e** $2x + 3y - 12 = 0$

Try to solve

3 Find the two equations of the tangent and the normal to the curve $x = \cos\theta$, $y = \sqrt{2} + \sin\theta$ at $\theta = \frac{\pi}{4}$

Critical thinking: if point (1 , 2) is one of the intersection points of the two curves: $y^2 - x^2 = 3$, $x\,y = 2$. Are the two tangents of the two curves perpendicular at this point? Explain



Important note : We say the two curves Z_1 , Z_2 intersect orthogonally if the two tangent drawn from their intersection point are perpendicular.

Exercises 1 - 4

1. If f , g and h are differentiable functions with respect to x , find the two equations of the tangent and the normal to the curve of the function h in each of the following using the values given in the opposite table :

   a. $h(x) = f(x) \times g(x)$, $x = 3$
   b. $h(x) = f(x) \div g(x)$, $x = 7$
   c. $h(x) = f[g(x)]$, $x = 3$

x	f(x)	g(x)	f' (x)	g' (x)
3	1	7	- 1	6
7	- 2	1	2	5

2. Find the two equations of the tangent and the normal to the curve of the function f where y = f(x) at the given values of x:

   a. $y = 3 - \cot^2 x$, $x = \frac{\pi}{4}$
   b. $y = 2 \cos x - \sec x$, $x = \frac{\pi}{3}$

3. Find the two equations of the tangent and the normal for each of the following curves at the given points :

   a. $x^2 + y^2 = 52$ at point (4 , - 6)
   b. $x^2 + 5xy + y^2 = 7$ at point (- 1 , - 1)
   c. $y^2 (1 + x^2) = 8$ at point (- 1 , 2)
   d. $(\sin x + \cos x)\, y = \cos^2 x$ at $x = \frac{\pi}{2}$

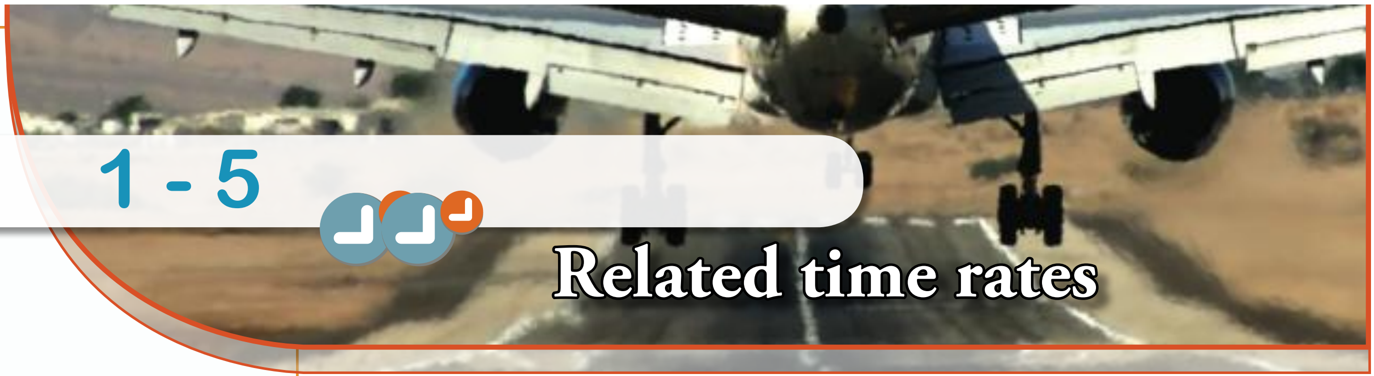
4. Find the two equations of the tangent and the normal for each of the following curves at the given points:

   a. $x = t^2 + 4t$, $y = 2t^2$ at $t = 1$
   b. $x = \sec\theta$, $y = \tan\theta$ at $\theta = \frac{\pi}{6}$

5. If point (4 , - 2) belongs to the curve $x^2 + y^2 - 2kx + 12 = 0$, find value of k, then find the equation of the tangent to the curve at this point.

6. **Area of triangle**: find the area of the triangle bounded by x-axis, tangent and the normal to the curve $x^2 + 4y^2 = 20$ at point (2 , 2)

7. **Orthogonal curves**: prove that the two curves $(x - 1)^2 + y^2 = 2$ and $(x + 1)^2 + y^2 = 2$ intersect orthogonally, then find the equations of their tangents at the intersection points.



1 - 5 Related time rates

You will learn

- The concept of the related time rates
- Methods of solvingthe equations of the related time rates
- Modeling and solving mathematical , physical and life problems

Key terms

- Rate
- Related Rates

Materials

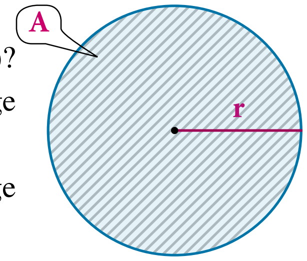
- Scientific calculator
- Computer graphics

Identifying and naming the variables
↓
Graphing the incomes
↓
Find the relation
↓
Differentating the relation with respect to time
↓
Substituting the values to find the required rate

Think and discuss

When a circular lamina is exposed to a heat source for a period of time (t) second

- Does its radius length (r) change over time (t)?
- Does the surface area (A) of the lamina change over time (t)?
- Does the surface area (A) of the lamina change as its radius length(r) changes? Explain.



Note that :

1 - The two variables a and r change overtime (a function of time) and they are related by the relation

$A = \pi r^2$ i.e : $A = f(r)$

2 - The differentiation of both sides of the relation above with respect to the time leads to a new equation relating between the related time rate of each variable and it is known as the equation of the related rates

Where: $\frac{dA}{dt} = f'(r) \times \frac{dr}{dt}$

3 - The time rate is positive if the variable increases by the time increasing and it is negative if the variable decreases by the time increasing.

Oral expression : Which of the following rates is positive?
(expansion - shrinking - approaching - diverge- molding- leaking - melting - accumulation - decreasing- increasing)

Example Inflating the balloon

1 When inflating a spherical balloon with gas, the rate of increase in its volume was 8π cm^3/sec when its radius length was 4cm. Find at this moment :

a The rate of increasing the radius length.

b The rate of increasing the surface area.

Solution

Let the volume of the balloon v, the radius length (r) and the surface area of the ballon (A) be differentiable functions at t.

a $V = \frac{4}{3}\pi r^3$ by differentiating both sides of the equation with respect to the time

$\frac{dv}{dt} = \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$ **(1)**

$\because \frac{dv}{dt} = 8\pi$ cm^3/sec , r = 4cm by substituting in the equation

$\therefore 8\pi = 4\pi(4)^2 \frac{dr}{dt}$ **i.e** $\frac{dr}{dt} = \frac{1}{8}$ cm/sec

b $A = 4\pi r^2$ by differentiating both sides of the equation with respect to the time

$\frac{dA}{dt} = 4\pi \times 2r \frac{dr}{dt} = 8\pi r \frac{dr}{dt}$ **(2)**

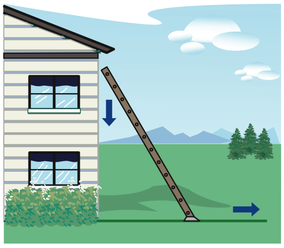
$\because \frac{dr}{dt} = \frac{1}{8}$ cm/sec , r = 4cm by substituting in the equation

$\therefore \frac{dA}{dt} = 8\pi \times 4 \times \frac{1}{8} = 4\pi$ cm^2/sec

Try to solve

1 **Volume:** A cube extends by heat so its edge length increases at a rate of 0,02 cm/min, and its surface area increases at a rate of 0,72 cm^2/min, at a moment . Find the cube edge length at this moment and the rate of the increase in its volume at this time .

Example Ladder motion

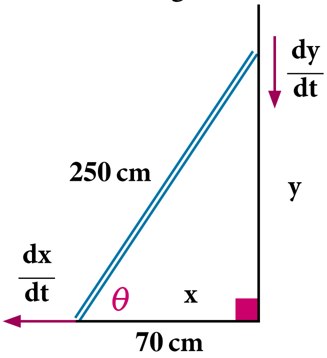


2 A 250cm ladder is leaning against a vertical wall . If the top of the ladder slid down at a rate of 10cm/sec when the base of the ladder is 70cm from the wall . Find:

a The rate of sliding the base of the ladder.

b The rate of change of the measure of the angle between the ladder and the ground.

Solution



a Let y be the distance between the top of the ladder and the ground and x the distance between the base of the ladder and the vertical wall,

From pythagorean theorem $x^2 + y^2 = (250)^2$ **(1)**

By differentiating both sides of the equation with respect to time

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ $\therefore \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$ **(2)**

$\because$ The top of the ladder slides down , then y decreases

$\therefore \frac{dy}{dt} = -10$ cm /sec

when x = 70 cm from equation **(1)** we find that: y = 240 cm

By substituting in equation **(2)** we deduce that: $\frac{dx}{dt} = -\frac{240}{7} \times -10 = \frac{240}{7}$ cm/sec

i.e the base of the ladder slides away from the wall at a rate of $\frac{240}{7}$ cm /sec

b Let θ be the measure of the angle of inclination of the ladder on the ground

$\sin\theta = \frac{y}{250}$ by differentiating both sides with respect to t

$\therefore \cos\theta \frac{d\theta}{dt} = \frac{1}{250}\frac{dy}{dt}$ but $\frac{dy}{dt} = -10$ cm/sec when x = 70 cm

$\frac{70}{250} \times \frac{d\theta}{dt} = \frac{1}{250} \times -10$ $\therefore \frac{d\theta}{dt} = -\frac{1}{7}$ rad/sec

i.e. The measure of the angle decreases at a rate of $\frac{1}{7}$ radian /sec

Try to solve

2 **ladder motion:** The base of a ladder is placed on horizontal ground and its top is leaning against a vertical wall. If the base of the ladder slid away from the wall at a rate of 30cm/sec , find the rate of sliding the top of the ladder when the measure of the angle between the ladder and the ground equals $\frac{\pi}{3}$

Critical thinking: A 15 ton rocket is launched and it emits the fuel at a constant rate 200 kg/s. What is the mass of the rocket after 30 second of the launching moment?

Important note: If $x_\circ$ is the initial value of variable x (when t = 0) , $\frac{dx}{dt}$ is the rate of change of x with respect to time and x is the value of the variable after time t , **then :** $x = x_\circ + \frac{dx}{dt} \times t$

in the previous "**critical thinking**" use the relation $k = k_\circ + \frac{dx}{dt} \times t$ to check your answer.

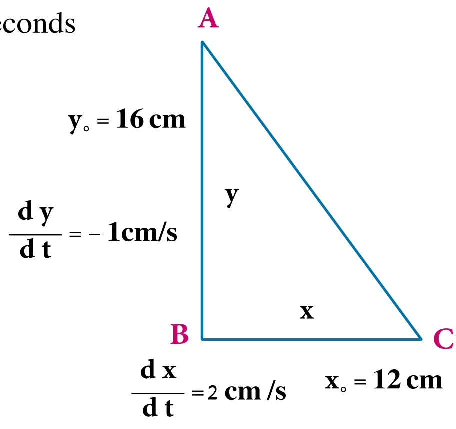
Example

Area

3 In a right - angled triangle, the lengths of the legs of the right angle are 12cm , 16cm. If the length of the first leg increases at a rate of 2 cm/s and the length of the second leg decreases at a rate of 1 cm/s.

a Find the rate of change of the triangle area after 2 seconds

b When is the triangle isosceles?



Solution

a Let x and y be the lengths of the legs after t second. the area of the triangle at this time where x, y and a are functions in the time:

$\therefore$ x = 12 + 2 t , y = 16 - t

$A = \frac{1}{2} x \times y = \frac{1}{2}(12 + 2n)(16 - t)$

A = (6 + t) (16 - t) by differentiating both sides of the equation with respect to time

$\therefore \frac{dA}{dt} = (6 + t) \times -1 + (16 - t) = 10 - 2t$ cm²/s

when t = 2 s $\therefore$ The rate of change of the triangle area = 10 - 2 (2) = 6 cm²/s

b When x = y then 12 + t = 16 - t $\therefore$ $t = \frac{4}{3}$ sec

i.e. after $\frac{4}{3}$ seconds, the right angled - triangle becomes isosceles triangle

Try to solve

3. **Volume:** A metal object in the form of a cuboid whose base is squared shape of side length increases at a rate of 1 cm/min and its height decreases at a rate of 2cm/min. Find the rate of increasing its volume when its base side length is 5 cm and its height is 20 cm. After how many minutes does the volume of the cuboid stop increasing?

Example

Length of shadow

4. A 1.8 meter man walks in a straight line approaching the base of a lamppost at a rate of 1.2 m/s if the height of the lamppost is 5.4 m on the ground , find :

a The rate of change of the length of the man's shadow.

b The rate of change of the man's head from the lamp when the man is 4.8 m from the lamppost.

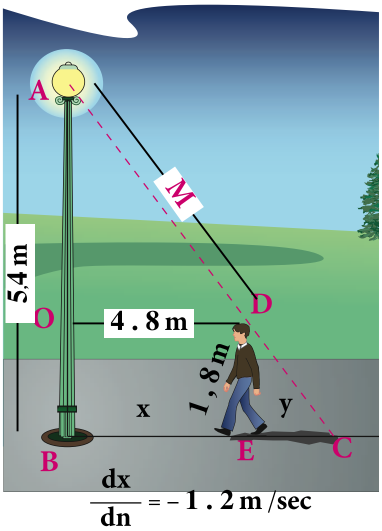
Solution

Modeling the problem: in the opposite figure $\overline{AB}$ represents the lamppost, point A is the lamp $\overline{DE}$ represents the man and point C is the end of the man's shadow , then :

x = EB the distance between the man and the base of the lamppost.

y = EC the length of the man's shadow.

M = AD the distance between the man's head and the lamp.



First: $\because \triangle ABC \sim \triangle DEC$

$\therefore \frac{AB}{DE} = \frac{BC}{EC} = \frac{5.4}{1.8} = \frac{x+y}{y}$

and 2 y = x by differentiating both sides of the equation with respect to time

$\therefore 2\frac{dy}{dt} = \frac{dx}{dt}$ i.e $\frac{dy}{dt} = \frac{-1{,}2}{2} = -0.6$ meter/s

Second: in the right angled - triangle A D O at (O)

$M^2 = x^2 + (3.6)^2$ by differentiating both sides of the equation with respect to time

$2M\frac{dM}{dt} = 2x\frac{dx}{dt}$ at x = 4,8 A f = 6 m

$6\frac{dM}{dt} = 4.8 \times -1.2$ **i.e.** $\frac{dM}{Dt} = -0.96$ meter/sec

Try to solve

4. **Constructions :** A 5 -meter water pipe with two ends A and B is leaning with its end A on a horizontal ground and with one of its points D against a 3-meter vertical wall. If end A slides away from the wall at a rate of $\frac{5}{4}$ meters/m, find the rate of sliding end B when the pipe reaches the edge of the wall.

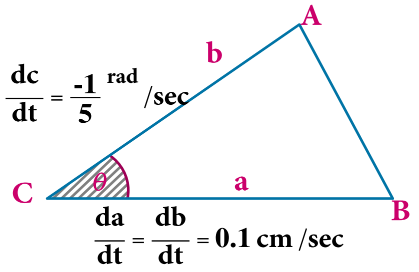
Example — Area

(5) Two sides in a triangle, the length of both increases at a rate of 0.1cm/sec and the measure of the included angle increases at a rate of $\frac{1}{5}$ rod/sec. At which rate does the area of the triangle change at the moment the length of each side of the triangle is 10cm?

Solution

Modeling the problem: at a certain moment t, let a side length of the triangle a the other side length b The measure of the included angle C^{rad} and A the area of triangle ABC be differentiable functions at t

where $A = \frac{1}{2}$ a b sin θ by differentiating both sides with respect to t



$$\therefore \frac{dA}{dt} = \frac{1}{2} a\, b \frac{d}{dt}[\sin c] + \frac{1}{2} \sin\theta \frac{d}{dt}[a\, b]$$

$$\frac{da}{dt} = \frac{1}{2} a\, b \cos c \frac{dc}{dt} + \frac{1}{2} \sin\theta [a \frac{db}{dt} + b \frac{da}{dt}] \qquad (1)$$

but $\frac{da}{dt} = \frac{db}{dt} = 0.1$, $\frac{dc}{dt} = \frac{1}{5}$

when the length of each side of the triangle is 10cm, the triangle is equilateral

then $m(\angle\theta) = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$ by substituting in equation **(1)**

$$\therefore \frac{dA}{dt} = \frac{1}{2} \times 10 \times 10 \times \frac{1}{2} \times \frac{1}{5} + \frac{1}{2} \times \frac{\sqrt{3}}{2} [2 \times 10 \times \frac{1}{10}]$$

$$= 5 + \frac{\sqrt{3}}{2} \simeq 5{,}866 \text{ cm}^2 /s$$

i.e. the area of the triangle increases at this moment at a rate of 5.866 cm^2 /sec

Try to solve

(5) **Area:** A B C is a right-angled triangle at C and its area is constant and equals $24cm^2$, if the rate of change of b equals 1cm/sec find the rate of change for each of a and m $(\angle A)$ at the moment in which b equals 8 cm.

Critical thinking: If x (the radian measure of an angle) increases at a constant time rate, explain why:

a) The sine and the tangent increase at the same rate when $x = 0$

b) The tangent increases at the rate equals 8 times the increase of sine when $x = \frac{\pi}{3}$

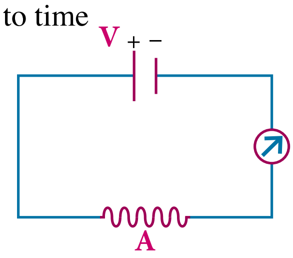
c) The cosine decreases at a rate of $\frac{3}{8}$ times the increase of the tangent when $x = \frac{\pi}{6}$

Example — Physics

(6) In a closed electric circuit, if v is the potential difference (volt), I is the current intensity (Ampere) R is the resistance (ohm). If the potential difference increases at a rate of 1 volt/s and the current intensity decreases at a rate of $\frac{1}{2}$ Ampere/s, find the rate of the resistance at the moment which v = 12 volt and I = 2 Amperes .

Solution

You know that V = I × R by differentiating both sides with respect to time



$\therefore \frac{dv}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$

$\because \frac{dv}{dt} = 1$ volt / sec , $\frac{dI}{dt} = -\frac{1}{2}$ Amperes /sec

$\therefore$ When v = 12 volts , I = 2 Amperes **then:** $R = \frac{V}{I} = \frac{12}{2} = 6$ ohm

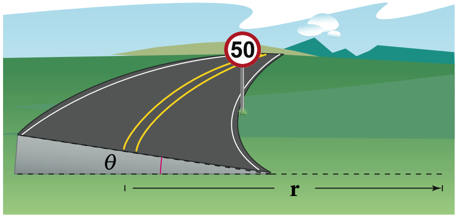
and $1 = 2 \times \frac{dR}{dt} + 6 \times -\frac{1}{2}$ $\quad \therefore \frac{dR}{dt} = 2$ ohm /sec

i.e the rate of the resistance at this moment is 2 ohm /s

Try to solve

6. in the previous example , calculate the rate of the resistance if the current increases at a rate of $\frac{1}{3}$ ampere / Sec.

Activity



Designing roads: in the ring roads and overhead bridges, we avoid the centrifugal force on the motion of the cars by designing the roads to incline on the horizontal plane by an angle of measure θ interior according to the relation: $v^2 = grtan\,\theta$ when (g) is the gravitational acceleration, V is the speed of the car (r) is the radius length of the curve circle . Find the relation equation between the rate of change of the car velocity and the rate of change of the road inclination .what would you advice car drivers to steer clear of the centrifugal force?

Exercises 1 - 5

Choose the correct answer :

1. If the radius length of a circle increases at a rate of $\frac{4}{\pi}$ cm/sec, then the circumference increases at this moment at the rate

   a) $\frac{4}{\pi}$ cm/s b) $\frac{\pi}{4}$ cm/s c) $\frac{1}{8}$ cm/s d) 8 cm/s

2. A cube of ice melts preserving its shape at a rate of $1 cm^3$/sec, then the rate of change of the cube edge length when its volume is 8 cm^3 is:.................. cm / s

   a) $-\frac{1}{12}$ b) $\frac{1}{12}$ c) $-\frac{1}{6}$ d) $\frac{1}{6}$

3. A body moves on the curve $y^2 = x^3$, if $\frac{dx}{dt} = \frac{1}{2}$ unit / sec when y = - 1, then $\frac{dy}{dx}$ at this moment equals .... unit / s

   a) $-\frac{3}{4}$ b) $-\frac{3}{8}$ c) $\frac{3}{4}$ d) $\frac{3}{2}$

4. If the slope of the tangent to the curve y = f(x) at a point = $\frac{1}{2}$ and the x-coordinate of this point decreases at a rate of 3 units / sec then the rate of change of its y-coordinates equals .... unit(s) / sec

   a) $-\frac{1}{6}$ b) $-\frac{3}{2}$ c) $\frac{1}{6}$ d) $\frac{3}{2}$

Answer the folowing:

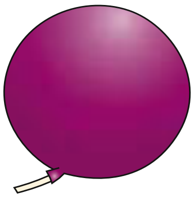
5 A point moves on a curve whose equation is $x^2 + y^2 - 4x + 8y - 6 = 0$. If the rate of change of its x-coordinate with respect to time at point (3, 1) equals 4 units / sec, find the rate of change of its y-coordinate with respect to time t .

6 A stone fell in a motionless lake and a circular wave was generated. If its radius length increases at a rate of 4cm/s, find the rate of increasing the surface area of the wave at the end of five seconds.

7 A regular hexagonal - like lamina shrinks by cold. if the rate of change of its side length is 0.1 cm/sec, find the rate of change in the area of the lamina when its side length is 10cm.

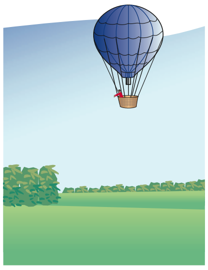
8 A known mass of gas with a constant temperature whose volume decreases at a constant rate of $2cm^3$/sec. If the pressure is inversely proportional to the volume and the pressure equals 1000 gm.wt /cm^2 when the volume is $250cm^3$, find the rate of change of the pressure with respect to time when the volume of gas is$100cm^3$.

9 If the gas leaks from a spherical balloon at a rate of $20cm^3$/sec, find the rate of change of the balloon radius length at the moment which the radius length is 10cm and find the rate of change of the balloon external surface area at the same moment.

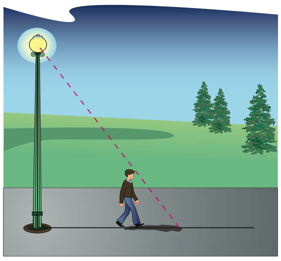


10 A 5-meter ladder is leaning against a vertical wall by its top and on a horizontal ground by its base. If the base of the ladder slides away from the wall at a rate of 4cm/min when the top of the ladder is 4 meter high of the ground , find the rate of sliding the top of the ladder , then find the rate of change of the measure of the angle between the ladder and the ground at this moment.

11 A ballon rises vertically from point A on the ground surface. An apparatus is placed to follow up the motion of the balloon at point B at the same horizontal plane of point A and distant 200 meters from it. A moment, the apparatus observed the elevation angle of the balloon to find it $\frac{\pi}{4}$, and it increases at a rate of 0.12 rad/min, find the rate of the balloon elevation at this moment.



12 A 180 cm man moves far from the base of a 3-meter lamppost at a rate of 1.2m/s, find the rate of change of the length of the man's shadow. If the straight line passing through the highest point of the man's head and the top of the lamp inclines on the ground with an angle of measure θ^{rad} when the man is far from the base of the lamppost for a distance x meter, prove that $x = \frac{6}{5} \cot \theta$, then find the rate of change of θ when the man is 3.6m from the base of the lamppost .



13 An isosceles triangle whose base length is $20\sqrt{3}$ cm . If the length of the two equal legs decreases at a rate of 3 cm/h , find the rate of decreasing the triangle surface area at the moment which the length of the two equal legs is equal to the base length .

Unit Summary

Derivatives of the trigonometric function

Function		Derivative
Sine function	$\sin x$	$\cos x$
Cosine function	$\cos x$	$-\sin x$
Tangent function	$\tan x$	$\sec^2 x$
Cotangent function	$\cot x$	$-\csc^2 x$
Secant function	$\sec x$	$\sec x \tan x$
Cosecant function	$\csc x$	$-\csc x \cot x$

Implicit Defferentiation

Differentiating the implicit relation $f(x, y) = 0$ requires to differentiate both sides of the relation with respect to one of the two variables x or y according to the chain rule to get $\frac{dy}{dx}$ or $\frac{dx}{dy}$ respectively .

Parametric Defferentiation

The curve given in the parametric form $x = f(t)$, $y = g(t)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ where f and g are two differentiable functions with respect to t

Higher - Derivatives of a function

If $y= f(x)$ where f is a differentiable function with respect to x, the derivatives starting from the second derivative (if found) are called the higher derivatives and they are denoted the symbol $\frac{d^2y}{dx^2}$ or y'' and the third derivative is denoted by the symbol y''' or $\frac{d^3y}{dx^3}$ and the n^{th} derivatives is denoted by the symbol $y^{(n)}$ or $\frac{d^n y}{dx^n}$ or $f^{(n)}(x)$ where n is a positive integer number.

The two equations of the tangent and the normal to a curve

If m the slope of the tangent to the curve $y = f(x)$ at the point (x_1 , y_1) which lies on it, then:

the equation of the tangent to the curve at point (x_1 , y_1) is : $y - y_1 = m (x - x_1)$

the equation of the normal at point (x_1 , y_1) is : $y - y_1 = - \frac{1}{m} (x - x_1)$

Related time Rates

If $y = f(x)$, x changes over time t , then y changes over time t. i.e. y is a composite function of the time t and $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ and this relation relates the time rate of change x with the time rate of change y .

The rate is positive if the variable increases with the increase of time.

The rate is negative if the variable decreases with the increase of time.

General Exercises

Choose the correct answer:

(1) If $y = 4 \sec^2 x$, then $y'(\frac{\pi}{4})$ equals:

(a) -8 (b) zero (c) $4\sqrt{2}$ (d) 16

(2) If $y = \sin 2x \cos 2x$, then $y''(\frac{\pi}{3})$ equals:

(a) -4 (b) zero (c) $4\sqrt{3}$ (d) 8

(3) A point moves on the curve $x y^2 = 12$. At point $(3, 2)$, then $\frac{dx}{dy}$ equals:

(a) -4 (b) $-\frac{3}{2}$ (c) $-\frac{1}{3}$ (d) 3

(4) If $y = 2t^3 + 7$ and $z = t^2 - 4$, then the rate of change of y with respect to z equals:

(a) $2t$ (b) $3t$ (c) 6 (d) 12

(5) The radius length of a circle increases at a rate of 2cm/min and its area at a rate of 20π cm^2/min, then the radius length of the circle at this moment equals: .... cm

(a) $\frac{5}{2}$ (b) 5 (c) 10 (d) 20

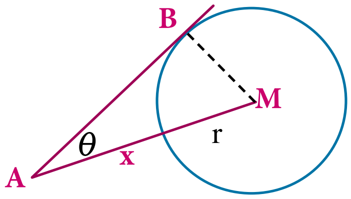
Answer the following:

(6) Find $\frac{dy}{dx}$ if y equals :

(a) $x + \cot 2x$ (b) $\sqrt[3]{x} + \sec 5x$ (c) $2x - 5\cos(\pi x)^2$

(d) $2\cos^2(\pi x + 1)$ (e) $\tan x \cot x$ (f) $\sec 2x \tan 2x$

(7) In the opposite figure: A is a point moving in the plane, $\overline{AB}$ is a tangent to circle M at B, AM = x + r where r is the radius length of the circle:



(a) Prove that $x = r(\csc\theta - 1)$

(b) Find the rate of change of x with respect to θ when $\theta = \frac{\pi}{6}$

(8) Find $\frac{dy}{dx}$ in the simplest for m each of the following:

(a) $x^2 - 3y^2 + 9 = 0$ (b) $5x^2 + 12y^3 - 7 = 0$

(c) $x^2 - 2xy + 2y^2 = 14$ (d) $(x-3)^2 + (y+2)^2 = 25$

(e) $xy + \sin x = 5$ (f) $\sin x \cos y = \frac{1}{2}$

(9) (a) Find the rate of change of $(x+3)(x-2)$ with respect to $\frac{x-3}{x+2}$

(b) If $f(x) = \frac{2}{x+1}$, $g(x) = 3x$

Find $\frac{d}{dx}[(f \circ g)(x)]$ when $x = -2$

10) a) If $y = \sqrt{2x+5}$, prove that $(2x+5)\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} = 0$

b) If : $xy = \sin ax$, prove that $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + a^2xy = 0$

c) If $y = a\cos x^2 + b\sin x^2$, prove that : $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = 0$

11) Find the two equations of the tangent and the normal to the following curves at the given point :

a) $x^2 - xy\sqrt{3} + 2y^2 = 12$ $(\sqrt{3}, 3)$

b) $x \sin 2y = y \cos 2x$ $(\frac{\pi}{4}, \frac{\pi}{2})$

12) Find the two equations of the tangent and the normal to the following curves at the given point:

a) $x = n^2 + 2n + 3$, $y = 2n^3 - 6n + 1$ at $n = 0$

b) $x = \sec^2\theta - 1$, $y = \tan\theta$ at $\theta = \frac{-\pi}{4}$

13) Find the area of the triangle bounded by y-axis , tangent and normal to the curve $4x^2 + y^2 = 20$ at point (1 , - 4).

14) Prove that the two curves $y^4 + 9y = 6x$ and $x^2 - 2y = 3x$ are intersected orthogonally at the origin point.

15) Prove that the curve $(\frac{x}{a})^n + (\frac{y}{b})^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ at point (a , b) regardless the value of n.

16) If a particle moves in a straight line and the relation between the distance and time is $s = 3t^3 + 3t^2 - 4$ where s is in centimeter and t is in second , find the rate of change of the distance with respect to time at the end of 3 seconds.

17) A spherical balloon is filled with gas but the gas leaks at a rate of x cm^3/sec , prove that the rate of the decrease of the balloon area at the moment which the radius length r cm equals $\frac{2x}{r}$ cm^2/sec.

18) A point moves on the curve $y^2 = 4x$. if the rate of change of its x-coordinate with respect to time at point (4, -4) equals 2 units/sec find the rate of change of its y-coordinate with respect to time.

19) The length of a rectangle is 24 cm and its width is 10cm. If its length decreases at a rate of 2cm/sec whereas its width increases at a rate of 1.5cm/sec, find the rate of change of its area after 4 seconds, then find the time at which the area stops increasing . What is the area of the rectangle then?

20) A constant length ladder with upper end sliding on a vertical wall at a rate of k units/sec, find the rate of sliding its base away from the wall when the ladder leans against the wall by an angle of θ where $\csc\theta = \frac{5}{4}$.

21. A metal regular quad pyramids whose height equals its base side length. Its volume increases at a rate of $1 cm^3/sec$, if the rate of the increase of both the pyramid's height and its base side length equals 0.01 cm/sec find its base side length .

22. A 2.6 meter ladder is leaning against a vertical wall with its top and its base on horizontal ground . If the base of the ladder slides away from the wall at a rate of 4m/min when the ladder is 1 meter from the wall, find the rate of the motion of the ladder top and the rate of change of the measure of the inclination angle of the ladder on the ground then.

23. A cuboid of dimensions 3,4 and 12cm. If the rate of the increase of its first dimension is 2cm/sec and second dimension is 1cm/sec but the rate of the decrease of its third dimension is 3cm/s, find the volume of the cuboid at any moment t and the rate of change of its volume at the end of two second .

24. Aright circular cylinder- like oil tank whose diameter length of its base is 24 meters. If we want to empty the tank from the oil at a rate of $2m^3/min$, what is the rate of the oil height in the tank?

25. A helicopter flies vertically at a fixed rate of 42m/min. If a ground observer is 150 m of the taking off site and observers the helicopter , find the rate of change of the elevation angle of the observer eyesight when the helicopter is 150 m high from the ground.

26. In a 100-meter race, the racer runs in a straight line toward the end line. If an end line camera is distant 5 meters and perpendicular to the racing pathway and at the same horizontal plane of the racers, find the rate of change of the angle by which the camera rotates to capture the racer when he was distant 5 meters from the end line and the rate of approaching the end point is 10m/sec.

27. If point A (x , y) moves on the curve of the function $y = x^3 + x$ where $\frac{dy}{dt} = 2$ unit / sec find the rate of change of the area of triangle A O B where (O) is the origin point and, B (0 , 6) at the moment at which x-coordinate of the moving point equals 3.

For more Activties and Excercises, Visit: www.sec3mathematics.com.eg

Accumulative test

Choose rhe correct answer:

1. If f(x) = cot x, then f'' $(\frac{\pi}{4})$ equals :

   a) $-\frac{4}{9}$ b) $\frac{4}{9}$ c) 4 d) $\frac{9}{2}$

2. A point moves on the curve $y^2 = 25 - x^2$ where $\frac{dx}{dt} = \frac{1}{2x+3}$, then at point (- 3 , 4), $\frac{dy}{dt}$ equals:

   a) $-\frac{1}{4}$ b) $\frac{1}{4}$ c) $\frac{1}{9}$ d) 9

3. If the equation of the normal to the curve y = f(x) at point (1 , 1) is x + 4 y = 5, then f'(1) equals:

   a) - 3 b) $-\frac{1}{4}$ c) 4 d) - 4

4. The tangent to the curve $y = 3x^2 - 5$ at point (1, - 2) passing through point:

   a) (5 , - 2) b) (3 , 1) c) (2 , - 4) d) (0 , - 8)

Answer the following questions:

5. If $x = t - t^2$, $y = t - t^3$, find $\frac{d^2y}{dx^2}$

6. If two parametric equations to a curve are $x = t^2 - 6t$, $y = 8\sqrt{t-2}$, find the equation of the tangent when t = 6

7. A B C is a triangle whose area is A and point C moves on the straight line y = 2 x, if A (L , 0) , B (0, k) where L and k are two positive constants, prove that $\frac{dA}{dx} = L + \frac{k}{2}$

8. Find the rate of change of $\sqrt{9 + x^2}$ with respect to $\frac{x}{x-1}$ when x = - 4

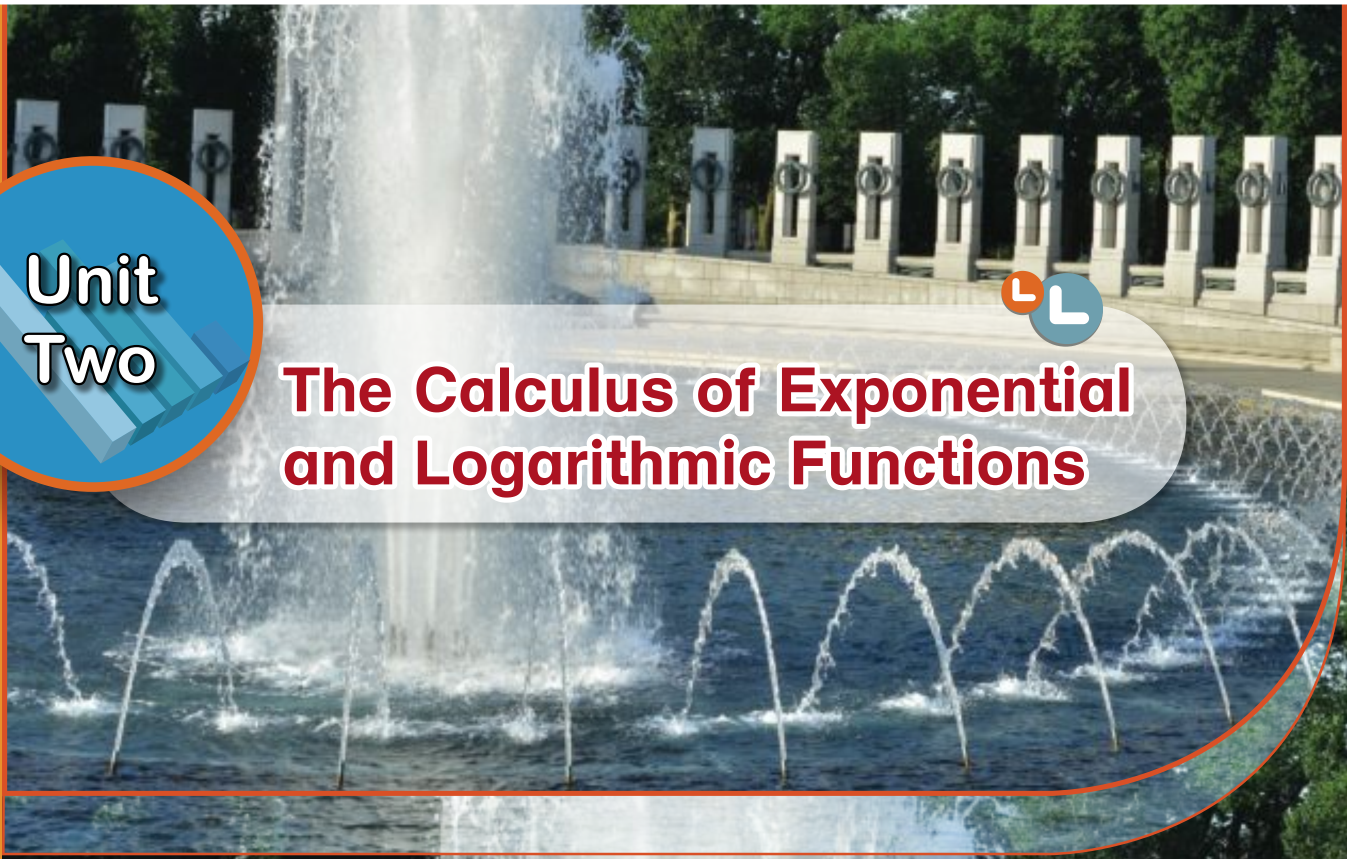
9. a) If $y = 4 + \cot x - \sec^2 x$, find the equation of normal when $x = \frac{\pi}{4}$

   b) A regular octagon whose side length is 10 cm and it increases at a rate of 0.2 cm/sec find the rate of the increase of its area.

10. A 4-meter ladder is leaning against a vertical wall with its top and its base on horizontal ground. If the base of the ladder slid away from the wall at a rate of 20cm/s, calculate the rate of sliding the top of the ladder down when the ladder is inclined on the ground with an angle measured $\frac{\pi}{3}$.

If you can't answer any of the questions above, you can use the next table :

Question No	1	2	3	4	5	6	7	8	9 A	9 B	10
Back to	3	2	4	4	2	2	2	2	4	5	5



Unit Two

The Calculus of Exponential and Logarithmic Functions

Introduction

In this unit, we are going to identify the Napier`s number e in regard to the scotch scientist John Napier (1550 - 1617) who had used the concept of logarithm in mathematics. It is also called the Eular's number in honor of the scientist who studied it and the functions relatied exhaustively. He had discovered the relation $e^{i\pi} + 1 = 0$ of the most five constants in mathematics which relate the trigonometric functions, exponential functions and the composite numbers.

The number e is a real irrational number equals 2.718281828459 approximatly. It is very important in mathematics where it is taken as a base to the exponential function with the natural base e^x [exp(x)] , and the natural logarithmic function $\log_e x$ [ln (x)]. You are going to learn each of these functions and their devivatives and also their antiderivatives (integration) using the graphic program to solve mathematical and life problems in different fields .

Unit objectives

By the end of the unit and carrying out the involved activities, the students should be able to:

- Identify the concept of the Napier`s number e through the limits.

$\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = e$, $\lim\limits_{x \to \infty} (1 + \frac{1}{x})^x = e$

- Find some limits that end to (approach to) the number e and its multiples

$\lim\limits_{x \to \infty} (1 + \frac{1}{x})^{2x} = \lim\limits_{x \to \infty} [(1 + \frac{1}{x})^x]^2 = e^2$

- Identify the concept of the natural logarithm $\log_e x$ through the limit $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \log_e x$

- Identify some properties of the natural logarithm such as:
  - $\log_e x = y \Leftrightarrow e^y = x$
  - $e^{\log_e x} = x$, $x > 0$
  - $\log_e x = 1$, $\log_a x = \frac{\log_e x}{\log_e a}$
- Find the derivatives of the exponential functions $y = e^x$, $y = a^x$, and the derivative of the logarithmic function $y = \log_e x$, $y = \log_a x$
- Identify the integration of the functions $y = e^x$ and $\log_e x$

Key terms

- Exponent
- Power
- Base
- Rational Exponents
- Exponential Growth
- Exponential Decay
- Exponential Function
- Exponential Equation
- Logarithm
- Form
- Common Logarithm
- Natural Logarithm
- Napier`s Constant
- Logarithmic Differentiation
- Antiderivative
- Integration
- Arbitrary constant
- Indefinite integral

Lessons of the unit

Lesson (2 - 1): The natural exponential function and the natural logarithmic function.

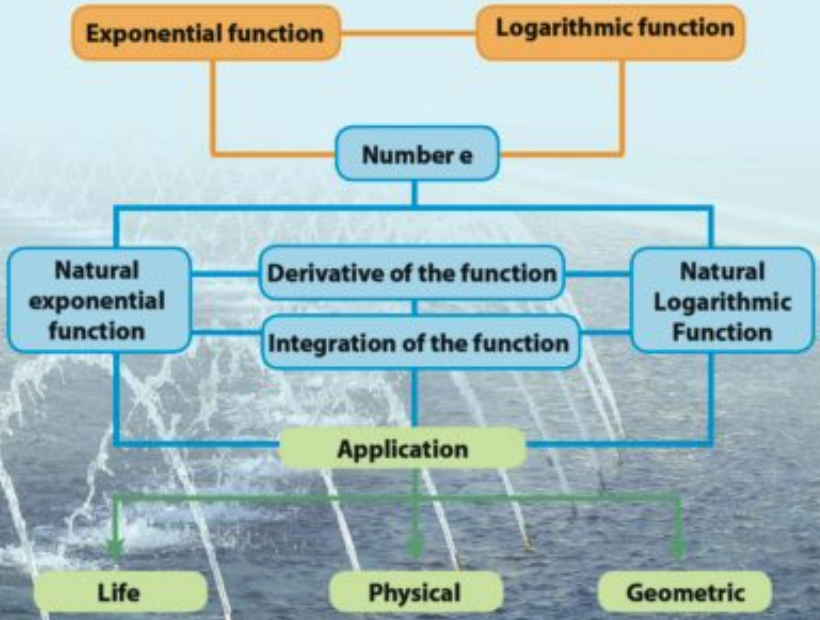
Lesson (2 - 2): Derivatives of the exponential and logarithmic functions.

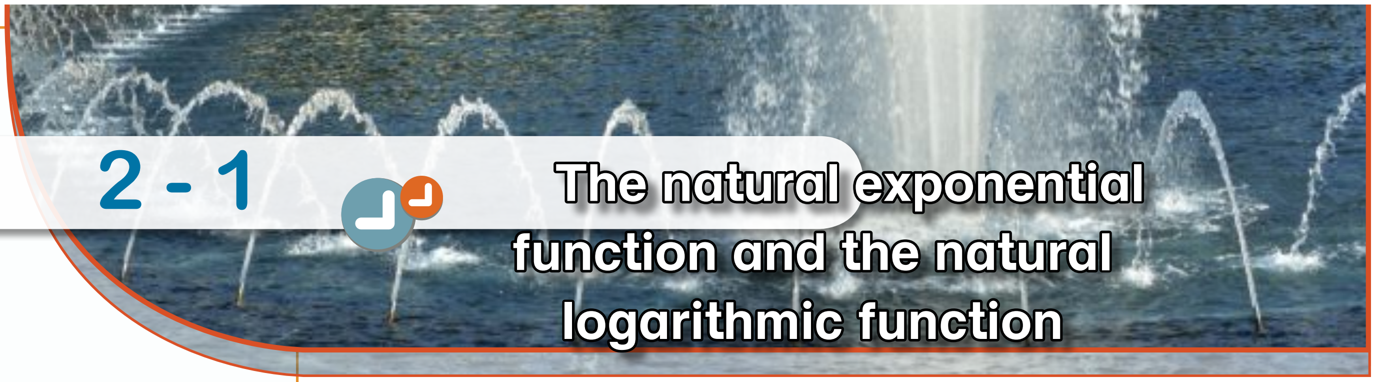
Lesson (2 - 3): Integration of the exponential and logarithmic functions.

Materials

- Scientific Calculator
- Computer graphics

Unit Planning guide





2 - 1 The natural exponential function and the natural logarithmic function

You will learn

- The concept of the Napier's number e through the limits.
- Finding the limit of a function approaches to the number e and its multiples.
- Define the natural exponential function .
- The concept of the natural logarithm through the Limits.

Key terms

- Natural exponential function
- Natural Logarithmic Functions

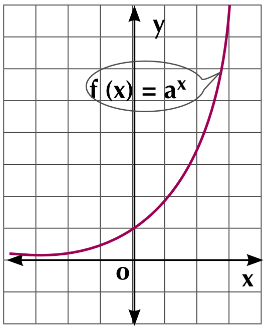
Materials

- Scientific calculator.
- Computer graphics.
- Internet

  Search for the number (e) in the internet to know more .

Explore:

You have previously learned the exponential function: $f(x) = a^x$ where $x \in \mathbb{R}$, $a \in \mathbb{R}^+ - \{1\}$ you have also known that the function curve passes through points $(0 , 1)$, $(1 , a)$, $(-1 , \frac{1}{a})$



- Do all the curves of the exponential function pass through point $(0 , 1)$? explain.
- What is the valve of the base a if the curve of the exponential function f passes through point $(1 , 3)$?

Definition

The number e

This number is known by the relation $e = \lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

- Draw the curve of the function f where $f(x) = e^x$ i.e f(x) = exp(x) using the geogebra program or any graphical program. Can you discover an approximate value for the number e?

Explore $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ use the calculator to complete the opposite table.

x	$(1 + \frac{1}{x})^x$
2	$(1 + \frac{1}{2})^2 = 2.25$
10	
100	
1000	
10000	
100000	

- Does this limit approach the approximate value previously determined for the number e? What do you infer?
- Does $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ equal $\lim\limits_{x \to 0} \left(1 + x\right)^{\frac{1}{x}}$? Explain

Note that

1) The value of e can be found using the calculator and pressing the buttons as follows:

Start → Shift ln 1 =

We find that $e \simeq 2.718281828$ to the nearest nine decimals 9

2) $\because e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ **(1)**

Let $m = \frac{1}{x}$ where $x \neq 0$ than $m \to 0$ when $x \to \infty$

$\therefore \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{m\to 0}(1+m)^{\frac{1}{m}}$

i.e. the number e can be expressed in the form of: $e = \lim_{x\to 0}\left(1+x^{\frac{1}{x}}\right)$

Example

Limits end to the powers of number e

1 Find:

a $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}$ **b** $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+3}$

Solution

a $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^3 = \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^3 = e^3$

b $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+3} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\left(1+\frac{1}{x}\right)^3$

$= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \times \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^3 = e \times (1+0)^3 = e$

Try to solve

1 Find:

a $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\frac{1}{5}x}$ **b** $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x+5}$

2 Find:

a $\lim_{x\to\infty}\left(1+\frac{5}{x}\right)^x$ **b** $\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^{x+4}$

Solution

a Let $y = \frac{5}{x}$ where $x \neq 0$ than $y \to 0$ when $x \to \infty$

$\therefore \lim_{x\to\infty}\left(1+\frac{5}{x}\right)^x = \lim_{y\to 0}(1+y)^{\frac{5}{y}} = \lim_{y\to 0}\left((1+y)^{\frac{1}{y}}\right)^5 = e^5$

b $\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^{x+4} = \lim_{x\to\infty}\left(\frac{x-1+3}{x-1}\right)^{x+4} = \lim_{x\to\infty}\left(1+\frac{3}{x-1}\right)^{x+4}$

$= \lim_{x\to\infty}\left(1+\frac{3}{x-1}\right)^{x-1} \times \lim_{x\to\infty}\left(1+\frac{3}{x-1}\right)^5 = e^3 \times (1)^5 = e^3$

Try to solve

2) Find:

a) $\lim\limits_{x \to \infty} \left(1 + \frac{3}{x}\right)^{x}$

b) $\lim\limits_{x \to \infty} \left(1 - \frac{1}{x}\right)^{2x}$

c) $\lim\limits_{x \to \infty} \left(\frac{x}{1 + x}\right)^{x}$

d) $\lim\limits_{x \to \infty} \left(\frac{2x + 5}{2x + 1}\right)^{x + 2}$

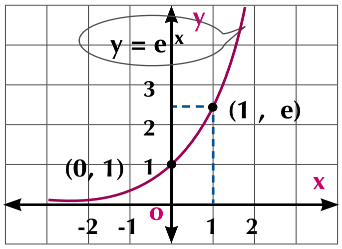
Note: The number e can be expressed using **(Taylor's series)** in the form of :

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots \infty = \sum_{n=0}^{\infty} \frac{1}{\lfloor n}$$

Learn

The natural exponential function

It is an exponential function whose base is e
$f(x) = e^x$, $x \in \mathbb{R}$



$\lim\limits_{x \to \infty} e^x = \infty$,

$\lim\limits_{x \to -\infty} e^x = 0$

Notice that

1) The domain of the function f where $f(x) = e^x$ is $\mathbb{R}$ and its range is $]0, \infty[$

2) The curve of the function passes through points (0 , 1) , (1 , e)

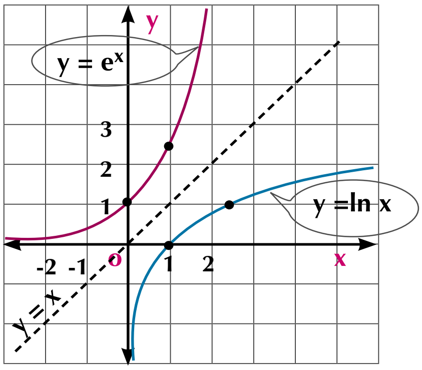
3) $f(x) = e^x$ is *One - to - One* function that accepts the existance of an inverse function called the natural logarithmic function.

4) The symbol *exp (x)* is used in any graphic program to graph the function

The natural logarithmic function

It is a logarithmic function whose base is e , $f(x) = \ln x$, $x \in \mathbb{R}^+$

Notice that:



$\lim\limits_{x \to \infty} \ln x = \infty$,

$\lim\limits_{x \to 0} \ln x = -\infty$

1) The domain of the function f where $f(x) = \ln x$ is $\mathbb{R}^+$ and its range is $\mathbb{R}$

2) The curve of the function passes through points (1 , 0) and (e , 1)

3) it is an inverse function for the function $y = e^x$

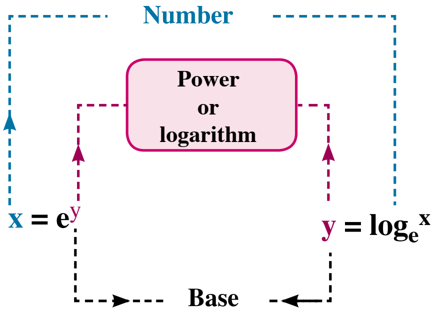
4) The symbol *ln(x)* is used in any graphic program to graph the function.

5) To find the value of ln 10 for example, press the following buttons:

Start → ln 1 0 =

We find that ln 10 = 2.302585093 to the nearest 9 decimals.

Some properties of the natural logarithm



The natural logarithm has the same logarithms properties which you have previously learned.

If $x \in \mathbb{R}^+$, $y \in \mathbb{R}$, $a \in \mathbb{R}^+ - \{1\}$ than:

1) The form $\ln x = y$ is equivalent to the form $e^y = x$

2) $e^{\ln x} = x$ **3)** $\ln e = 1$

4) $\ln 1 = 0$ **5)** $\log_a x = \dfrac{\ln x}{\ln a}$ **(the property of changing the base)**

For each x and $y \in \mathbb{R}^+$ and $n \in \mathbb{R}$

6) $\ln xy = \ln x + \ln y$ **7)** $\ln \dfrac{x}{y} = \ln x - \ln y$

8) $\ln x^n = n \ln x$ **9)** $\log_e x \times \log_x e = 1$

Note that the natural logarithms can be used to do different calculations by the same way of using the common logarithms but it needs more efforts especially $\ln 10 \simeq 2.3026$. As a result, it is favorable to use the natural logarithm to solve problems related to limits, differentiation and the exponential and logarithmic equations with the base e

Limits and natural logarithm

Example

(3) Prove that $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \ln a$ where $a > 0$

Solution

Let: $y = a^x - 1$, when $x \to 0$ then $y \to 0$ **(1)**

$a^x = 1 + y$ **by taking the natural logarithm of both sides**

$\therefore \ln a^x = \ln(1 + y)$ **by using the property of power logarithm**

$\therefore x \ln a = \ln(1 + y)$ **i.e:** $x = \dfrac{\ln(1 + y)}{\ln a}$ **(2)**

From **(1)** and **(2)** we deduce:

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y}{\ln(1 + y)} \times \ln a = \lim_{y \to 0} \frac{\ln a}{\frac{1}{y} \ln(1 + y)} = \lim_{y \to 0} \frac{\ln a}{\ln(1 + y)^{\frac{1}{y}}}$$

$$= \frac{\ln a}{\lim\limits_{y \to 0} \ln(1 + y)^{\frac{1}{y}}} = \frac{\ln a}{\ln \lim\limits_{y \to 0} (1 + y)^{\frac{1}{y}}} = \frac{\ln a}{\ln e} = \ln a$$

Important note: **The previous example is used as a rule to solve the problems. You can also use the following limits to solve the problems:**

(1) $\lim\limits_{x \to 0} \dfrac{\log_a (1 + x)}{x} = \log_a e$ **(2)** $\lim\limits_{x \to 0} \dfrac{\ln (1 + x)}{x} = 1$

Try to solve

3. Prove that: $\lim\limits_{n \to \infty} n \left[\ln (n + 1) - \ln n\right] = 1$

Exercises 2 - 1

Choose the correct answer:

1. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{2x}$ equals:
   - a) 1
   - b) 2
   - c) e
   - d) e^2
2. $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{3x}}$ equals :
   - a) $\dfrac{1}{3}$
   - b) e^3
   - c) $e^{\frac{1}{3}}$
   - d) $\dfrac{e}{3}$
3. $\lim\limits_{x \to 0} \dfrac{2^x - 1}{3x}$ equals:
   - a) $3 \ln 2$
   - b) $\dfrac{1}{3} \ln 2$
   - c) $\ln \dfrac{2}{3}$
   - d) $2 \ln 3$
4. $\lim\limits_{x \to 1} \dfrac{\ln x}{x - 1}$ equals:
   - a) 0
   - b) 1
   - c) e
   - d) e^{-1}

Find:

5. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{x + 1}$
6. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x + 1}\right)^{x}$
7. $\lim\limits_{m \to \infty} \left(1 + \dfrac{1}{m}\right)^{m x}$
8. $\lim\limits_{x \to \infty} \left(\dfrac{x + 7}{x + 3}\right)^{x + 4}$
9. $\lim\limits_{x \to 0} \dfrac{\ln (1 + 2x)}{x}$
10. $\lim\limits_{x \to 0} \dfrac{\ln (1 + x^2)}{x^2}$

Find the following limits:

11. $\lim\limits_{x \to \infty} \left(1 + \dfrac{4}{x}\right)^{x + 3}$
12. $\lim\limits_{x \to \infty} \dfrac{e^{-x} - 1}{x}$
13. $\lim\limits_{x \to 0} \dfrac{a^{2x} - 1}{x}$
14. $\lim\limits_{x \to 0} (1 + 3 \tan^2 x)^{\cot^2 x}$
15. $\lim\limits_{x \to \infty} \left(\dfrac{2x - 1}{2x + 1}\right)^{x}$
16. $\lim\limits_{x \to 0} \dfrac{\ln (1 + 3x^2)}{2x^2}$

Derivatives of Exponential and Logarithmic Functions

2 - 2

Explore:

Use the calculator to complete the next table and explore $\lim\limits_{x \to 0} \frac{e^x - 1}{x}$

x	.001	.0001	.00001	0	-0.00001	-0.0001	$-.001$
$\frac{e^x - 1}{x}$							.999500

Revise example (4) in the previous lesson and check your exploration.

Learn

Derivative of the natural exponential function

if $f(x) = e^x$ then $f'(x) = e^x$

From the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x (e^h - 1)}{h}$$

$$= e^x \times \lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) = e^x \times 1 = e^x$$

i.e $\frac{d}{dx}(e^x) = e^x$

Example — Derivative of the natural exponential function

1. Find the first derivative for each of:

a $y = x^2 + 3e^x$ **b** $y = x^3 e^x$ **c** $y = \frac{2e^x}{x+1}$

Solution

a $\because y = x^2 + 3e^x \quad \therefore \frac{dy}{dx} = 2x + 3\frac{d}{dx}(e^x) = 2x + 3e^x$

b $\because y = x^3 e^x$

$$\therefore \frac{dy}{dx} = x^3 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^3)$$

$$= x^3 e^x + 3x^2 e^x = x^2 e^x (x + 3)$$

You will learn

- Derivatives of exponential functions.
- Derivatives of logarithmic functions.
- Logarithmic differentiation.
- Higher derivatives of exponential and logarithmic functions.
- Modeling the problems.

Key terms

- Derivative
- Chain Rule
- First Derivative
- Logarithmic Differentiation

Materials

- Scientific calculator.

c $\because y = \dfrac{2e^x}{x+1}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{(x+1)\dfrac{d}{dx}(2e^x) - 2e^x \dfrac{d}{dx}(x+1)}{(x+1)^2} = \dfrac{2xe^x}{(x+1)^2}$

Try to solve

1 Find $\dfrac{dy}{dx}$ for each of the following:

a $y = 2e^x + \cos 2x$ **b** $y = e^x \sin x$ **c** $y = \dfrac{e^x}{\tan x}$

Critical thinking: What is the relation between the slope of the tangent to the curve $y = e^x$ at any point on it and the y - coordinate of this point? Explain

Chain rule

If Z is a differentiable function of x, $f(z) = e^z$

then: $\dfrac{d}{dx}(e^z) = e^z \cdot \dfrac{dz}{dx}$

Example

2 Find the first derivative for each of:

a $y = e^{3x^2+5}$ **b** $y = 3e^{\sec x}$ **c** $y = (e^{3x} - e^{-2x})^5$

Solution

a $\because y = e^{3x^2+5}$ $\quad \therefore \dfrac{dy}{dx} = e^{3x^2+5} \times \dfrac{d}{dx}(3x^2+5) = 6x\, e^{3x^2+5}$

b $\because y = 3e^{\sec x}$ $\quad \therefore \dfrac{dy}{dx} = 3\, e^{\sec x} \times \dfrac{d}{dx}(\sec x) = 3\, e^{\sec x} \cdot \sec x \tan x$

c $\because y = (e^{3x} - e^{-2x})^5$ $\quad \therefore \dfrac{dy}{dx} = 5\,(e^{3x} - e^{-2x})^4\,[3e^{3x} + 2e^{-2x}]$

Try to solve

2 Find $\dfrac{dy}{dx}$ for each of the following:

a $y = 2x + e^{6x}$ **b** $y = \frac{1}{2} e^{7-x^2}$ **c** $y = (e^{2x} + e^{-2x})^3$

Learn

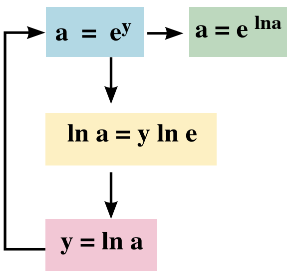
Derivative of exponential function to the base a

If $f(x) = a^x$ **than** $f'(x) = a^x \ln a$

Notice that $a = e^{\ln a}$ **(from the logarithm properties)**

$\therefore a^x = \left[e^{\ln a}\right]^x = e^{x\ln a}$

and $\dfrac{d}{dx}(a^x) = \dfrac{d}{dx}(e^{x\ln a}) = e^{x\ln a} \times \ln a = a^x \times \ln a$



in general: $\frac{d}{dx}(a^z) = a^z \log_e a \cdot \frac{dz}{dx}$

Example **The derivative of the exponential function**

(3) Find $\frac{dy}{dx}$ for each of the following:

a $y = 5 \times 6^x$ b $y = 3^{(3x^2 - 5x + 2)}$ c $y = e^{\sin x} \times 2^{-5x}$

Solution

a $\because y = 5 \times 6^x$ $\therefore \frac{dy}{dx} = 5 \frac{d}{dx}(6^x) = 5 \times 6^x \ln 6$

b $\because y = 3^{3x^2 - 5x + 2}$ $\therefore \frac{dy}{dx} = (6x - 5) \times 3^{(3x^2 - 5x + 2)} \times \ln 3$

c $\because y = e^{\sin x} \times 2^{-5x}$ $\therefore \frac{dy}{dx} = e^{\sin x} \frac{d}{dx}(2^{-5x}) + 2^{(-5x)} \frac{d}{dx} e^{\sin x}$

$= e^{\sin x}[-5 \times 2^{-5x} \ln 2] + 2^{-5x}[e^{\sin x} \cos x]$

$= 2^{-5x} e^{\sin x}[\ln 2^{-5} + \cos x]$

Try to solve

(3) Find $\frac{dy}{dx}$ for each of the following:

a $y = 5^{x^2 + 2x}$ b $y = 2^{\sec^2 x}$ c $y = e^{2x} a^{x^2 - 5}$

Learn

Derivative of natural logarithm function

if $f(x) = \ln x,\ x > 0$ **then** $f'(x) = \frac{1}{x}$

Notice that the logarithmic function is an inverse function of the exponential function

If $y = \ln x$ **then** $x = e^y$ **(1)**

by differentiating both sides of relation **(1)** with respect x $\therefore 1 = e^y \frac{dy}{dx}$ **(2)**

From **(1)** and **(2)** : we de deduce that $\frac{dy}{dx} = \frac{1}{x}$ **I.e:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Example **Derivative of natural logarithm function**

(4) Find the first derivative for each of the following:

a $y = 3x + \ln x$ b $y = (2x^5 - 3) \ln x$ c $y = \frac{\ln x - 1}{\ln x + 1}$

Solution

a $\because y = 3x + \ln x$ $\quad \therefore \frac{dy}{dx} = 3 + \frac{d}{dx}(\ln x) = 3 + \frac{1}{x}$

b $\because y = (2x^5 - 3) \ln x$ $\quad \therefore \frac{dy}{dx} = (2x^5 - 3)\frac{d}{dx}(\ln x) + (\ln x)\frac{d}{dx}(2x^5 - 3)$

$= (2x^5 - 3) \times \frac{1}{x} + 10x^4 \ln x$

$= \frac{1}{x}[(2x^5 - 3) + 10x^5 \ln x]$

c $\because y = \frac{\ln x - 1}{\ln x + 1}$ $\quad \therefore \frac{dy}{dx} = \frac{(\ln x + 1) \times \frac{1}{x} - (\ln x - 1) \times \frac{1}{x}}{(\ln x + 1)^2} = \frac{2}{x(\ln x + 1)^2}$

Try to solve

4 Find $\frac{dy}{dx}$ for each of the following:

a $y = 5 - 3 \ln x$ **b** $y = x^2 \ln x$ **c** $y = \frac{1 - 2\ln x}{\ln x}$

Critical thinking: What is the relation between the slope of tangent to the curve y =ln x at any point on it and the x-coordinate to the tangent point? Explain.

Chain rule

- **If** z is a differentiable function with respect to x and f(z) =ln z **than:** $\frac{d}{dx}[\ln z] = \frac{1}{z} \bullet \frac{dz}{dx}$

- **If** $x < 0$, **then:** $\frac{d}{dx}[\ln(-x)] = \frac{1}{-x} \times -1 = \frac{1}{x}$

- **In general** $\frac{d}{dx}[\ln |x|] = \frac{1}{x}$ for each $x \neq 0$

, $\frac{d}{dx}[\log_e |z|] = \frac{z'}{z}$ where z is a differentiable function at x.

Example

5 Find $\frac{d}{dx}$ for each of the following:

a $y = \ln(2x^3 + 9)$ **b** $y = x^4 \ln x^3$ **c** $y = \ln \frac{x^2}{x + 7}$

Solution

a $\because y = \ln(2x^3 + 9)$ $\quad \therefore \frac{dy}{dx} = \frac{1}{2x^3 + 9} \times \frac{d}{dx}(2x^3 + 9) = \frac{6x}{2x^3 + 9}$

b $\because y = x^4 \ln x^3$ $\quad \therefore \frac{dy}{dx} = x^4 \frac{d}{dx}(\ln x^3) + \ln x^3 \frac{d}{dx}(x^4)$

$$= x^4 \times \frac{1}{x^3} \times 3x^2 + \ln x^3 \times 4x^3$$

$$= 3x^3 + 4x^3 \ln x^3 = x^3 [3 + 4 \ln x^3]$$

c $\frac{d}{dx}\left(\ln \frac{x^2}{x+7}\right) = \frac{x+7}{x^2} \times \frac{(x+7)(2x) - x^2 \times 1}{(x+7)^2} = \frac{x+14}{x(x+7)}$

Try to solve

5 Find $\frac{dy}{dx}$ for each of the following:

a $y = \ln (7x - 3)^2$ **b** $y = 2x^2 \ln x^3$ **c** $y = \frac{x}{\ln x}$

Learn

Derivative of logarithmic function to the base a

$$\text{If } f(x) = \log_a x \quad \text{, then } f'(x) = \frac{1}{x \log_e a}$$

Notice $\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a} \frac{d}{dx}[\ln x] = \frac{1}{x} \cdot \frac{1}{\ln a}$

and $\frac{d}{dx}(\log_a x) = \frac{1}{x} \log_a e$

in general $\frac{d}{dx}(\log_a z) = \frac{1}{z \ln a} \cdot \frac{dz}{dx}$

Remember

From the properties of logarithms

$\log_a x = \frac{\ln x}{\ln a}$

$\log_e a \times \log_a e = 1$

Derivative of the logarithmic function

Example

6 Find $\frac{dy}{dx}$ for each of the following:

a $y = \log_3 x$ **b** $y = \log_5 (3x - 2)$ **c** $y = \log (2x - 3)^2$

Solution

a $\because y = \log_3 x$ $\therefore \frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\ln 3} = \frac{1}{x \ln 3}$

b $\because y = \log_5 (3x - 2)$ $\therefore \frac{dy}{dx} = \frac{3}{(3x - 2) \ln 5}$

c $\because y = 2\log|2x - 3|$ $\quad \therefore \frac{dy}{dx} = \frac{2 \times 2}{(2x - 3)\ln 10} = \frac{4}{(2x - 3)\ln 10}$

Try to solve

6 Find the slope of the tangent for each of the following curves at the given values of x:

a $y = \log_2 5x \quad , \quad x = 2$ **b** $y = 4\log(3x + 1) \quad , \quad x = 1$

c $y = \log_2 (2x^2 - 3)^4 \quad , \quad x = 1$ **d** $y = 3(\log x)^2 \quad , \quad x = 3$

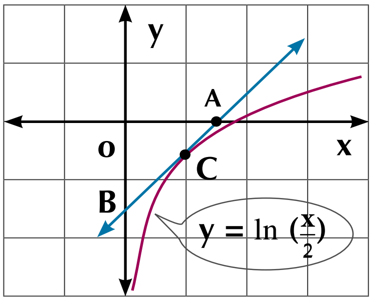
7 Geometric applications: If $\overleftrightarrow{AB}$ is a tangent to the curve $y = \ln\frac{x}{2}$ at point C (1 , y) and intersects x - axis at point A and y-axis at point B, find the length of $\overline{AB}$

Solution

To find the length of $\overline{AB}$, follow the opposite graph:

length of $\overline{AB}$ ← intersection point A, B ← equation of tangent $\overleftrightarrow{AB}$

The slope of the tangent at any point: $\frac{dy}{dx} = \frac{1 \times 2}{x} \times \frac{1}{2} = \frac{1}{x}$



$\because \overleftrightarrow{AB}$ touches the curve at any point C (1 , y)

then $y = \ln\frac{1}{2} = -\ln 2$ i.e C (1 , - ln 2), at which

$\frac{dy}{dx} = 1$, and the equation of the tangent $\overleftrightarrow{AB}$ at C is:

$y + \ln 2 = x - 1$

$\because \overleftrightarrow{AB}$ intersects X -axis at point A $\quad \therefore A(1 + \ln 2 , 0)$

and y - axis at point B $\quad \therefore B(0 , -1 - \ln 2)$

Then $(AB)^2 = (1 + \ln 2)^2 + (1 + \ln 2)^2$ $\quad \therefore AB = \sqrt{2}(1 + \ln 2)$

Try to solve

7 If the normal to the curve $y = \ln 2x$ at point A (1 ,ln 2) intersects x-axis at point B, find the length of $\overline{AB}$ to the nearest three decimals.

Mathematical Application

Logarithmic differentiation

The relation among the variables can be expressed in a logarithmic form by taking off the natural logarithm of its both sides and using the properties of logarithms to simplify the relation before doing the differentiation operations.

Example

8 Find $\frac{dy}{dx}$ for each of the following:

a $y = (x^3 + 5)^x$ **b** $y = [\sin x]^{\tan x}$

Solution

a $\because\ y = (x^3 + 5)^x$ — By taking the natural logarithm of both sides of the relation

$\therefore\ \ln y = x \ln (x^3 + 5)$ — By differentiating both sides of the relation with respect to x

$\frac{1}{y}\frac{dy}{dx} = \ln (x^3 + 5) + \frac{x}{x^3 + 5} \times 3x^2$ — by multiplying both sides $\times\ y = (x^3 + 5)^x$

$\therefore\ \frac{dy}{dx} = (x^3 + 5)^x \left[\frac{3x^3}{x^3 + 5} + \ln (x^3 + 5)\right]$

b $\because\ y = [\sin x]^{\tan x}$ — By taking the natural logarithm of both sides of the relation

$\ln y = \tan x \ln \sin x$ — By differentiating both sides of the relation with respect to x

$\frac{1}{y}\frac{dy}{dx} = \tan x \times \frac{d}{dx}(\ln \sin x) + \ln \sin x \times \frac{d}{dx}(\tan x)$

$= \frac{\sin x}{\cos x} \times \frac{1}{\sin x} \times \cos x + \ln \sin x \times \sec^2 x$

$= 1 + \sec^2 x \log_e \sin x$ — by multiplying both sides $\times\ y = [\sin x]^{\tan x}$

$\therefore\ \frac{dy}{dx} = [\sin x]^{\tan x} (1 + \sec^2 x \ln \sin x)$

Try to solve

8 Find $\frac{dy}{dx}$ for each of the following

a $y = x^{2x}$ **b** $y = (\sin x)^x$ **c** $y^2 = 3^x \times 2^y$

9 Satisfying a relation: If $y = e^{-x}\sqrt{\frac{1 + x}{1 - x}}$ where $-1 < x < 1$, prove that: $(1 - x^2)\, y' = x^2 y$

Solution

$\because\ y = e^{-x}\sqrt{\frac{1 + x}{1 - x}}$ — By taking the logarithm of both sides to base e

$\therefore\ \ln y = \ln e^{-x} + \frac{1}{2}\ln \frac{1 + x}{1 - x}$

$\ln y = -x + \frac{1}{2}[\ln (1 + x) - \ln (1 - x)]$

By differentiating both sides of the relation with respect to x

$\frac{1}{y} \times y' = -1 + \frac{1}{2}\left[\frac{1}{1 + x} - \frac{-1}{1 - x}\right]$

$\frac{y'}{y} = -1 + \frac{1}{2}\left[\frac{1 - x + 1 + x}{1 - x^2}\right]$

$\frac{y'}{y} = -1 + \frac{1}{1 - x^2} = \frac{-1 + x^2 + 1}{1 - x^2}$

$\frac{y'}{y} = \frac{x^2}{1 - x^2}$ $\therefore\ (1 - x^2)\, y' = x^2 y$

Try to solve

9) If $y = ae^{\frac{b}{x}}$, prove that: $x\,y\,y'' + 2\,y\,y' - x\,y'^2 = 0$

Exercises 2 - 2

Choose the correct answer

1) If $f(x) = e^{3x}$, then $f'(x)$ equals:

a) e^{3x} b) $3e^{3x}$ c) $9e^{3x}$ d) $3e^{2x}$

2) If $f(x) = ae^{x}$, then $f'(-2)$ equals:

a) $-f(2)$ b) $-f'(2)$ c) $-f(-2)$ d) $f(-2)$

3) The curve of the function f: $f(x) = 1 + \ln(x - 2)$ is the same curve of the function g : $g(x) = \ln x$ by translation:

a) $(1, 2)$ b) $(1, -2)$ c) $(-2, 1)$ d) $(2, 1)$

4) The ratio between the slope of the tangent to the curve $y = \ln 3\sqrt{x+1}$ and the slope of the tangent to the cure $y = \ln 5\sqrt{x+1}$ when $x = a$ is

a) $3 : 5$ b) $5 : 3$ c) $1 : 1$ d) $\ln 3 : \ln 5$

Find the first derivative for each of the following:

5) $y = e^{3x^5}$

6) $y = e^{x^2 - x}$

7) $y = (3^{x-1})^{-2}$

8) $y^2 = e^{5x^2 - 3}$

9) $y = \ln(2x - 7)$

10) $y = \ln(\frac{1}{2}x^2 + x)$

11) $y = \ln \frac{x^2}{x+7}$

12) $y = x^2 \ln x$

13) $y = \log_3 (4x + 9)^2$

14) $y = \frac{e^{3x}}{\log x}$

15) $y = \sec e^{x}$

16) $y = 2e^{3x} - 5\log_2 \frac{x}{5}$

Find the slope of the tangent for each of the following curves at the given values:

17) $y = \sqrt{x} - 2e^{x}$, $x = \frac{1}{4}$

18) $y = x^2 - 3\ln x$, $x = 2$

19) $y = \frac{1}{4}e^{2x} - 2\ln x$, $x = \frac{1}{2}$

Find $\frac{dy}{dx}$ for each of the following:

20) $y e^x = e^3$ 21) $x \ln y = 58$ 22) $y = x^{\sin x}$

23) $y = e^{e^x}$ 24) $y = e^{x^e}$ 25) $y = x^{\frac{1}{x}}$

Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ for each of the following:

26) $x = e^{2n}$, $y = n^3$

27) $x = 6 \ln n$, $y = n^2$

Answer the following:

28) If $y = x^2 \ln \frac{x}{a}$, find $\frac{d^3y}{dx^3}$ when $x = 4$

29) If $e^x = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor n}$ (Taylor's expansion), prove that: $\frac{d}{dx}(e^x) = e^x$

30) If $y = \sqrt{\frac{x^2 + 1}{x^2 - 1}}$, prove that $(x^4 - 1) y' + 2x y = 0$

31) Find the values of x at which the tangent of the curve $y = 9x^3 - 8\ln x$ is parallel to x-axis.

32) Find the equation of the normal to the curve $y = 3e^x$ at a point lying on it and its x-coordinate equals -1

33) Industry: If the daily production of a factory during a certain period of time t (day) is identified by the relation $y = 400 (1 - e^{-0.30})^t$ units, find the rate of change in the number of the produced units with respect to time on the tenth day.

34) Life application: If the production of a beehive is given by the relation: $y = (n + 100) \ln (t + 5)$ grams in terms of the number of days t, find the rate of change of the beehive production when $t = 5$, $t = 15$, $t = 20$. Does the production of the beehive increase or decrease?

2 - 3 Integrals of Exponential and Logarithmic Function

You will learn

- Integration of exponential and logarithmic functions.
- Geometric applications.
- Physical applications.

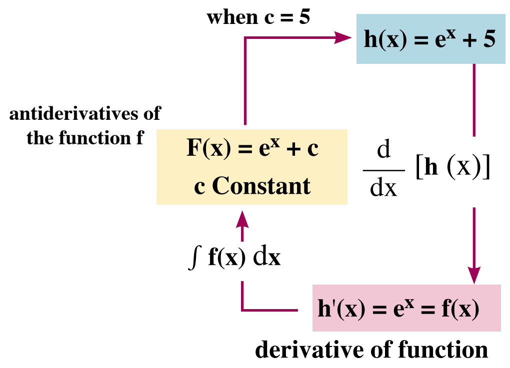
Key terms

- Antiderivative
- Integration
- Indefinite integral
- Arbitrary constant

Materials

- Scientific calculator.
- Computer graphics.

Explore



From your previous learning to differentiation, you knew that the derivative of the function h with respect to x where $h(x) = e^x + 5$ is $h'(x) = e^x$

If we denote the function $f'(x)$ by the symbol f(x), then we can use an inverse operation (indefinite integration) to find an indefinite number of other functions(F(x) + c) the derivative of each equals f(x) and called the set of antiderivatives of the function f one of them equals h(x) where:

$\int f(x)\,dx = F(x) + c$ where c is an arbitrary

Explore the set of the antiderivatives for each of:

$f(x) = 5e^{5x}$, $g(x) = 8e^{x^4}$, $g(x) = \frac{1}{x}$

Learn

Indefinite integrals of exponential function

> **If K is a real number where $k \neq 0$**
>
> **Then:** $\int e^x\,dx = e^x + c$
>
> , $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$ where c is an arbitrary constant

Example

1 Find:

a $\int e^{7x}\,dx$ b $\int e^{-\frac{3}{4}y}\,dy$ c $\int 8e^{2z}\,dz$

Solution

a $\int e^{7x}\,dx = \frac{1}{7}e^{7x} + c$

b $\int e^{-\frac{3}{4}y}\, dy = \frac{1}{-\frac{3}{4}} e^{-\frac{3}{4}y} + c = \frac{-4}{3} e^{-\frac{3}{4}y} + c$

c $\int 8e^{2z}\, dz = 8\int e^{2z}\, dz = \frac{8}{2} e^{2z} + c = 4e^{2z} + c$

Remember

$\int a\, f(x)\, dx = a \int f(x)\, dx$

Try to solve

1 Find

a $\int \pi e^{x}\, dx$ **b** $\int -e^{-5z}\, dz$ **c** $\int -6e^{0.2y}\, dy$ **d** $\int 3\sqrt{2}\, e^{-\sqrt{2}\,n}\, dn$

Example

2 Find each of the following integrations:

a $\int \frac{e^{x} + e^{-x}}{2}\, dx$ **b** $\int \frac{3e^{x} - 2e^{2x}}{2e^{x}}\, dx$

Solution

a $\int \frac{e^{x} + e^{-x}}{2}\, dx = \frac{1}{2}\int (e^{x} + e^{-x})\, dx$

$= \frac{1}{2}\left[\int (e^{x}\, dx + \int e^{-x}\, dx\right]$

$= \frac{1}{2}(e^{x} - e^{-x}) + c$

b $\int \frac{3e^{x} - 2e^{2x}}{2e^{x}}\, dx = \int \frac{3}{2}\, dx - \int e^{x}\, dx$

$= \frac{3}{2}x - e^{x} + c$

Remember

$\int [f(x) \pm g(x)]\, dx$

$= \int f(x)\, dx \pm \int g(x)\, dx$

Try to solve

2 Find:

a $\int \frac{e^{x} + e^{-x}}{e^{x}}\, dx$ **b** $\int (x^{2} + 2e^{x})\, dx$ **c** $\int (x^{2e} + e^{3x})\, dx$

Notice that: **If** f(x) is a differentiable function, then:

$$\int e^{f(x)} \cdot f'(x)\, dx = e^{f(x)} + c$$

Example

3 **a** $\int \sin x\, e^{\cos x}\, dx$ **b** $\int 4x\, e^{x^{2}+1}\, dx$

Solution

a By putting $f(x) = \cos x$ $\therefore f'(x) = -\sin x$

$\int \sin x\, e^{\cos x}\, dx = -\int e^{\cos x}(-\sin x)\, dx = -e^{\cos x} + c$

b By putting $f(x) = x^2 + 1$ $\therefore f'(x) = 2x$

$\int 4x\, e^{x^2+1}\, dx = 2\int e^{x^2+1}(2x)\, dx = 2e^{x^2+1} + c$

Try to solve

3 Find the following integrations:

a $\int (\cos x\, e^{\sin x} + 3x^2)\, dx$

b $\int (x - 3)\, e^{x^2 - 6x + 5}\, dx$

Indefinite integral of logarithmic functions

You know that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ where $x > 0$, $\frac{d}{dx}(\ln - x) = \frac{1}{x}$ where $x < 0$

In general $\frac{d}{dx}\ln|x| = \frac{1}{x}$ where $x \neq 0$

I.e the function $\ln|x|$ where $x \neq 0$ is one of the antiderivatives of the function $\frac{1}{x}$

So:

$$\int \frac{1}{x}\, dx = \ln|x| + c \qquad \text{where } x \neq 0$$

Multiples of the function

Example

4 Find each of the following integrations:

a $\int \frac{2}{x}\, dx$ **b** $\int \frac{7}{x \ln 3}\, dx$

Solution

a $\int \frac{2}{x}\, dx = 2\int \frac{1}{x}\, dx = 2\ln|x| + c$ where $x \neq 0$

b $\int \frac{7}{x\ln 3}\, dx = \frac{7}{\ln 3}\int \frac{1}{x}\, dx = \frac{7}{\ln 3}\ln|x| + c$ where $x \neq 0$

Try to solve

4 Find:

a $\int \frac{\ln 3}{x}\, dx$ **b** $\int \frac{4}{3x \ln 5}\, dx$ **c** $\int \frac{\ln x^2}{x \ln x^3}\, dx$

Example

5 Find each of the following integrations:

a $\int \left(3x^2 + \frac{5}{x}\right) dx$ b $\int \left(\frac{2e}{x} + \frac{x}{e}\right) dx$ c $\int \frac{(3x-1)^2}{3x} dx$

Solution

a $\int \left(3x^2 + \frac{5}{x}\right) dx = \int 3x^2\, dx + \int \frac{5}{x} dx = x^3 + 5 \ln |x| + c$

b $\int \left(\frac{2e}{x} - \frac{x}{e}\right) dx = 2e \int \frac{1}{x} dx - \frac{1}{e} \int x\, dx = 2e \ln |x| - \frac{x^2}{2e} + c$

c $\int \frac{(3x-1)^2}{3x} dx = \int \frac{9x^2 - 6x + 1}{3x} dx = \int \left(3x - 2 + \frac{1}{3x}\right) dx$

$= \frac{3x^2}{2} - 2x + \frac{1}{3} \ln |x| + c$ where $x \neq 0$

Try to solve

5 Find each of the following integrations:

a $\int \frac{6x^2 - 5}{3x} dx$ b $\int \frac{x^2 - 4}{x^2 - 2x} dx$ c $\left(\sqrt{x} - \frac{3}{\sqrt{x}}\right)^2 dx$

Notice that: If the function is differentiable and , $f(x) \neq 0$, then

$$\int \frac{1}{f(x)} \cdot f'(x)\, dx = \ln |f(x)| + c$$

Example

6 Find each of the following integrations:

a $\int \frac{4}{1 + 2x} dx$ b $\int \frac{2x + 3}{x^2 + 3x - 2} dx$ c $\int \tan x\, dx$

Solution

a $\because (1 + 2x)' = 2$ $\therefore \int \frac{4}{1 + 2x} dx = 2 \int \frac{(1 + 2x)'}{1 + 2x} dx = 2 \log_e |1 + 2x| + c$

b $\because (x^2 + 3x - 2)' = 2x + 3$

$\therefore \int \frac{2x + 3}{x^2 + 3x - 2} dx = \ln |x^2 + 3x - 2| + c$

c $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\ln |\cos x| + c = \ln |\sec x| + c$

Try to solve

6 Find each of the following integrations:

a $\int \cot x\, dx$ b $\int \frac{x^2 - 4}{x^2 + 2x} dx$ c $\int \frac{(x^2 + 2)\, dx}{x^3 + 6x + 1}$

Example

(7) Geometric applications: the slope of the tangent to a curve at any point (x , y) on it equals $\frac{3x + 2}{x}$. Find the equation of the curve if known that it passes through point (e , 3e + 5)

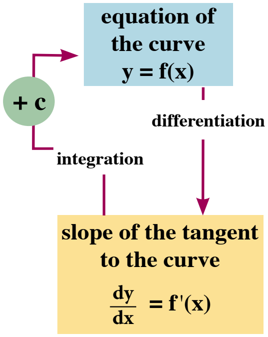
Solution

Let the equation of the curve y = f(x)

$\because$ the slope of the tangent at any point $= \frac{dy}{dx} = \frac{3x + 2}{x}$

$\therefore y = \int \frac{dy}{dx}\, d x = \int (3 + \frac{2}{x})\, d x$

$\therefore y = 3x + 2 \ln x + c$ where c is an arbitrary constant



$\because$ The curve passes through point (e , 3e + 5), then it satisfies its equation: i.e.

$3e + 5 = 3 (e) + 2 \ln e + c$ $\therefore c = 3$

The equation of the curve is: $y = 3x + 2 \ln x + 3$

Try to solve

(7) The slope of the tangent to the curve of the function f at any point (x , y) on it equals $\frac{1}{2x - e}$ and $f(e) = \frac{1}{2}$. Find f(2e)

Example

(8) Physical applications: If the rate of change in the surface area of a lamina A (in square centimeters) with respect to time t (second) is identified by the relation $\frac{dA}{dt} = e^{-0.1t}$ and the area of the lamina at the beginning of the change equals 80 cm^2, find the surface area of the lamina after 10 second.

Solution

Surface area of the lamina $A = \int \frac{dA}{dt}\, d t = \int e^{-0.1t}\, dt$

$\therefore A = -10e^{-0.1t} + c$

At the beginning of change t = 0 , A = 80 $\therefore c = 90$

The surface area of the lamina at any moment $A = 90 - 10e^{-0.1t}$

After 10 second $\therefore$ the surface area of the lamina $= 90 - 10e^{-1}$ cm^2

Try to solve

(8) If the rate of change of the sales in a factory is inversely proportional to time in weeks and the sales of the factory after 2 weeks and 4 weeks were 200 units and 300 units respectively, find the sales of the factory after 8 weeks.

Exercises 2 - 3

Choose the correct answer:

1. If $f''(x) = \frac{1}{2}[e^{x} + e^{-x}]$, $f(0) = 1$, $f'(0) = 0$, then f(x) equals:
   a) $-f'(x)$ b) $f'(x)$ c) $-f''(x)$ d) $f''(x)$

2. If the slope of the tangent to a curve at any point (x , y) equals $4e^{2x}$, $f(0) = 2$, then f(-2) equals:
   a) 4 b) $4e^{-4}$ c) $2e^{-4}$ d) 2e

3. $\int \tan\theta \, d\theta$ equals
   a) $-\ln|\cos\theta| + c$ b) $-\ln\cos\theta + c$ c) $\ln\cos\theta + c$ d) $|\ln\cos\theta| + c$

4. $\int 4x\, e^{x^2}\, dx$ equals
   a) $\frac{1}{2} e^{x^2} + c$ b) $e^{x^2} + c$ c) $2e^{x^2} + c$ d) $4e^{x^2} + c$

Find each of the following integrations:

5. $\int e^{4x}\, dx$
6. $\int (3x^2 + 2e^{x})\, dx$
7. $\int (\frac{4}{x} - e^{-x})\, dx$
8. $\int e^{1-3x}\, dx$
9. $\int \frac{7}{3} e^{3x-4}\, dx$
10. $\int 2e^{x}(e^{x}+1)^2\, dx$
11. $\int \frac{e^{3x} + 2e^{2x} + 4}{e^{x}}\, dx$
12. $\int x^2 e^{x^3+1}\, dx$
13. $\int \frac{2e^{x}}{e^{x}+1}\, dx$
14. $\int \frac{dx}{4x-1}$
15. $\int \frac{x}{x^2+1}\, dx$
16. $\int \frac{\sec^2 x}{\tan x}\, dx$
17. $\int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx$
18. $\int \frac{\cos x}{1 + \sin x}\, dx$
19. $\int \frac{\sec x \tan x}{\sec x - 1}\, dx$
20. $\int \frac{1}{(x \ln x)}\, dx$
21. $\int \frac{2x}{(x+1)^2}\, dx$
22. $\int \frac{(\log x)^2}{x}\, dx$
23. $\int \frac{3x^2}{x^3 - 1}\, dx$
24. $\int \frac{3x^2 - 5}{x^3 - 5x + 1}\, dx$
25. $\int \frac{4e^{x} + x e^{2x}}{x e^{x}}\, dx$
26. $\int \frac{4}{x \ln 3x}\, dx$
27. $\int \frac{(1 + \ln x)^2}{x}\, dx$

28. Geometric applications: If the slope of the tangent to the curve of the function f at any point (x , y) equals $2e^{-\frac{1}{2}x}$ and $f(0) = 1$, find f(3)

Unit Summary

The number e is known by the relation

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \quad , \quad e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}$$

The exponential function with the natural base is a function whose base is e where $f(x) = e^x \ , \ x \in \mathbb{R}$

The natural logarithmic function is a logarithmic function whose base is e where $f(x) = \ln x$ and $x \in \mathbb{R}$

Derivatives of exponential and logarithmic functions

Function	Derivative	Condition
e^x	e^x	$x \in \mathbb{R}$
$e^{f(x)}$	$e^{f(x)} \bullet f'(x)$	f is differentiable
a^x	$a^x \ln a$	$a > 0 \ , \ a \neq 1$
$\ln \lvert x \rvert$	$\frac{1}{x}$	$x \neq 0$
$\ln \lvert f(x) \rvert$	$\frac{1}{f(x)} \bullet f'(x)$	f is differentiable , $f(x) \neq 0$

Integration of exponential and logarithmic functions

Function	Derivative	Condition
e^x	$e^x + c$	$x \in \mathbb{R}$
e^{kx}	$\frac{1}{k} e^{kx} + c$	$k \neq 0$
$e^{f(x)} \bullet f'(x)$	$e^{f(x)} + c$	f differentiable
$\frac{1}{x}$	$\ln \lvert x \rvert + c$	$x \neq 0$
$\frac{1}{f(x)} \bullet f'(x)$	$\ln \lvert f(x) \rvert + c$	f differentiable , $f(x) \neq 0$

General Exercises

Choose the correct answer:

1. The curve of the function f where $f(x) = e^{x-3} + 2$ is the same curve of $g(x) = e^x$ by translation:
   - **a** (2 , 3)
   - **b** (2 , -3)
   - **c** (3 , -2)
   - **d** (3 , 2)

2. If $f'(x) = x f(x)$, $f(3) = -5$, then: $f''(3)$ equals:
   - **a** -50
   - **b** -40
   - **c** 15
   - **d** 27

3. $\int \frac{\ln x^2}{\ln x} dx$ equals:
   - **a** $\frac{x}{2} + c$
   - **b** $\frac{1}{x} + c$
   - **c** $2x + c$
   - **d** $\log_e |x| + c$

4. $\int \frac{1}{x \ln x^3} dx$ equals:
   - **a** $3\ln |x| + c$
   - **b** $3\ln |\ln x| + c$
   - **c** $\frac{1}{3} \ln |x| + c$
   - **d** $\frac{1}{3} \ln |\ln x| + c$

Find the solution set to the following equations:

5. $\log (x - 3) = 2$
6. $\log_2 (x + 7) = 5$
7. $e^{5x+1} = 20$
8. $\ln 4^{2x-1} = \ln 50$
9. $\ln e^x = 2 \ln 3$
10. $7^x = 25$

Find each of the following limits:

11. $\lim_{n \to \infty} (1 + \frac{2}{n})^{3n}$
12. $\lim_{x \to \infty} (\frac{n}{n+1})^n$
13. $\lim_{x \to \infty} (\frac{2x-1}{2x+1})^{2x}$

Find $\frac{dy}{dx}$ for each of:

14. $y = 7e^x - 3$
15. $y = e^{3x^2+1}$
16. $y = x^2 e^{-3}$
17. $y = \ln (x^2 + 3)$
18. $y = x \ln x^2$
19. $y = e^x \ln (x^2 + 1)$
20. $y = \frac{e^x}{e^x + 1}$
21. $y = x^2 \ln e^x$
22. $y = \frac{1}{x} \ln e^x$

Find $\frac{d^2 y}{d x^2}$ for each of the following:

23. $x = e^{2n}$, $y = e^{3n}$
24. $x = n^3$, $y = 4 \ln n$

Find $\frac{d^3y}{dx^3}$ for each of the following:

25. $y = \ln x$
26. $y = x^3 + e^{2x}$
27. $y = x \log x$

28 If $e^{xy} = x^2 + y$, prove that: $(x e^{xy} - 1) y' = 2x - y e^{xy}$

29 If $x^2 y = a b \ln x$, prove that: $x^2 y'' + 5x y' + 4y = 0$

Find $\frac{dy}{dx}$ for each of the following:

30 $y = 2^x (x + 1)$

31 $y = 3x^{x^2} + 3$

32 $y = (1 - 3x)^{\cos x}$

33 $y = x^{e^x}$

Find the values of x at which the tangent of the curve is parallel to x-axis where $x > 0$

34 $y = x^3 \ln x$

35 $y = \frac{1}{x^2} \ln x^2$

36 $y = x^3 - 81 \ln x$

37 $y = 3 - \frac{x^2}{4} + \ln \frac{x}{2}$

38 If $x e^{-\frac{1}{2}y} + y e^{-\frac{x}{2}} = 2$, find $\frac{dy}{dx}$ when $x = 0$

Find each of the following integrations:

39 $\int \frac{9 + 6x}{x^2 + 3x} \, dx$

40 $\int \frac{\ln x}{x \ln \sqrt{x}} \, dx$

41 $\int (\frac{3}{x} + 4e^{-2x}) \, dx$

42 $\int (\frac{e^3}{x} + x \ln 3) \, dx$

43 $\int (x^3 - \frac{2}{3} e^x) \, dx$

44 $\int \cot^3 x \, dx$

45 **Intersection with axes:** if the tangent of the curve $y = e^x$ at point $(2, e^2)$ intersects x-axis at point A and y-axis at point B, find the length of $\overline{AB}$

46 **The two equations of the tangent and the normal:** Find the two equations of the tangent and the normal of the curve $y = x^3 - 18 \ln x$ at a point lies on it and its x-coordinate equals 2.

47 **Inverse proportion:** if the slope of the tangent at any point (x, y) on the curve of the function f is inversely proportional to x and the slope of the tangent equals 2 when $x = 4$ and $y = 2$, find y in terms of x.

48 **Parallel:** Find the values of x (to the nearest two decimals) at which the tangent of the curve $y = \frac{1}{x^2} \ln x^2$ is parallel to x-axis.

For more activities and exercise, visit the following site www.sec3mathematics.com.eg

Accumulative test

Choose the correct answer:

1. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{2x}$ equals:

   a) 1   b) e   c) e^2 d) e^{-2}

2. $m = e^{\ln 7}$, then m equals:

   a) 1   b) $\frac{1}{e}$ c) e   d) 7

3. The solution set of the equation ln (x - 3) + ln (x - 2) = ln 6 equals:

   a) {0 , 5}   b) {5}   c) {2 , 3}   d) ϕ

4. If $f(x) = x^2 - 3 \ln 5x$, then f' (2) equals:

   a) -1   b) 1   c) $\frac{5}{2}$ d) 6

Answer the following Questions:

5. Find the first derivative for each of:

   a) $y = (x^3 + 1)^2$ b) $y = \ln x^2 - e^{\frac{1}{4}x}$ c) $y = \ln\left[\frac{e^{x^2}}{x^2}\right]$

6. If $y = e^{3x} + x^2$, prove that : $\frac{d^2y}{dx^2} - 2 = 9(y - x^2)$

7. Find each of the following integrations:

   a) $\int \frac{2x^2 + 3}{x} \, dx$ b) $\int \frac{dx}{\sqrt{x}\, e^{\sqrt{x}}}$ c) $\int \frac{3}{2x \ln x} \, dx$

8. If the slope of the tangent to the curve of the function at any point (x , y) on it equals $7 - 2e^x$ and f(ln 2) = 3 , find f(x).

If you cannot answer any of these questions, you can use the following table:

Question no	1	2	3	4	5	6	7	8
back to	1	1	1	2	2	2	3	3

Unit Three

Behavior of the Function and Curve Sketching

Introduction

Through reading the graph to a curve of a function , you can determine monotonic intervals [increasing-decreasing -constancy]. You can also know the maximum and minimum values of the function and identify some properties of the function. You can use computer graphics to graph and study the behavior of the function. But it is not usually available. In this unit , you will identify more techniques to graph the curve of the function through differentiation using the function derivatives (first derivative and second derivative) to determine the interval of increasing or decreasing the function. You can also determine the maximum and minimum values related to the values of x (the maximum values and local minimum values) and the absolute maximum and minimum values of a function continuous on a limited interval [a, b]. You can also determine the convexity direction to the curve of the function (upward or downward) you will learn some applications to find the maximum and minimum values to help you model and solve mathematical physical and life problems.

Unit objectives

By the end of the unit and carrying out the involved activities, the students should be able to:

- Use the first derivative to study the increase and decrease of the differentiable function.
- Identify the local minimum and maximum values of the differentiable function
- Identify and find the absolute minimum and absolute maximum values of a function in a closed interval
- Find the critical points، convex upward، convex downward and inflection points of a function.
- Find the relation between the curve of the function and the first derivative
- Learn the behavior of the function in regard to the monotony، maximum values and minimum values throughout the first derivative
- Sketch the curves of the polynomial functions up to the third degree only.

Key terms

- Increasing Function
- Dereasing Functicn
- Maximum and minimum values
- Maximum values
- Critical Point
- Local Minimum value
- Local Maximum value
- Absolute Extrema value
- Convexity
- Convex Upward
- Convex Downward
- Inflection Point

Lessons of the unit

Lesson (3 - 1): Increasing and decreasing the functions.

Lesson (3 - 2): maximum and minimum values

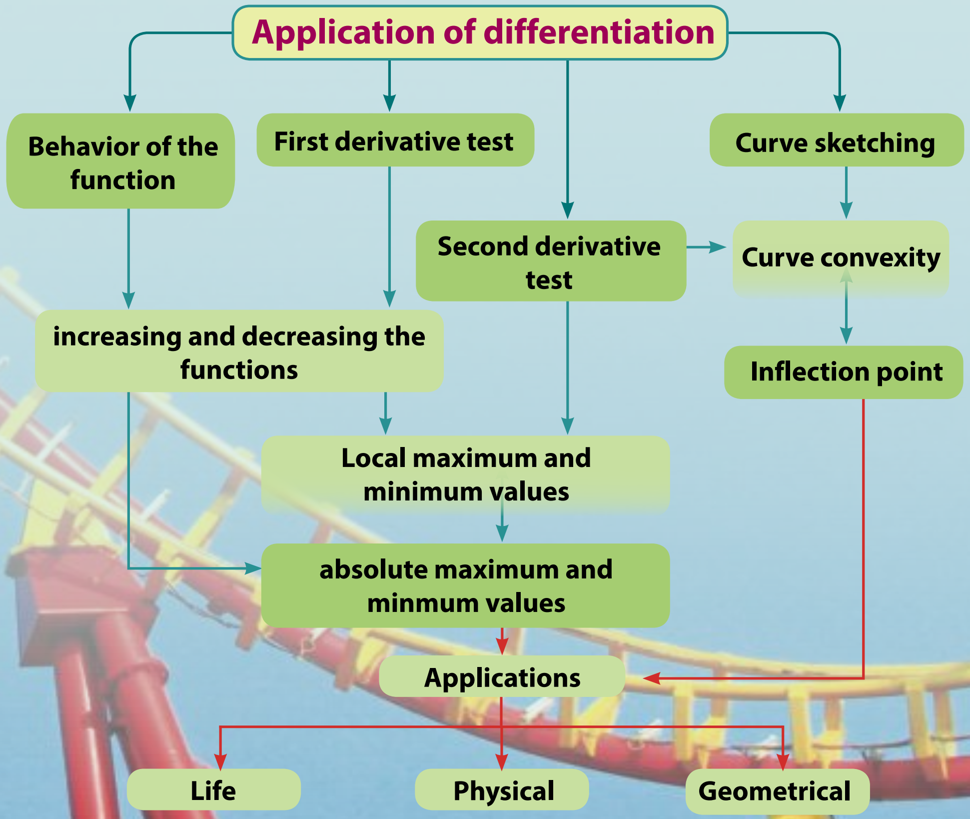
Lesson (3 - 3): Curve sketching

Lesson (3 - 4): Applications on maximum and minimum values

Materials

- Scientific calculator
- Computer graphics

Unit planning guide



3 - 1 Increasing and Decreasing Functions

You will learn

- Using the first derivative to determine the increasing or decreasing intervals of a function.
- Daily life applications on the increasing and decreasing intervals of the functiuons

Key terms

- Increasing Function
- Decreasing Function

Materials

- Scientific calculator
- Computer graphics

Investigating monotony of a function

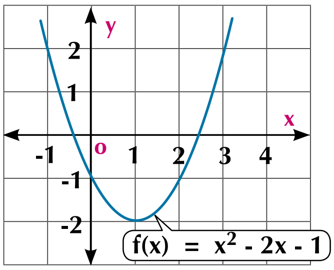
Find f ' (x) → solve equation f ' (x) = 0 → investigate the sign of f ' (x) → f ' (x) < 0 : f decreasing ; f ' (x) > 0 : f increasing

Think and discuss

The opposite figures show the two curves of the functions f , g where

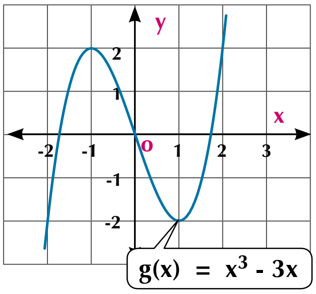
f (x) = x^2 - 2x - 1 ,

g (x) = x^3 - 3x



- Determine the increasing and decreasing intervals of the function f
- Find the derivative of function f and investigate the sign of f ' (x) for the different values of x which belong to the increasing interval
- Investigate the sign of f ' (x) for the different values of x which belong to the decreasing interval

Repeat the steps above to determine the sign of g' (x) in the increasing and decreasing intervals of the function g. What do you infer? what kind of angles does the tangent of the curve make at the different values of x in the increasing intervals with the positive direction of x-axis?



Learn

First derivative test for the monotonic functions

Theorem

Let f be a differentiable function on the interval] a , b [:

1- If f ' (x) > 0 for all the values of x ∈] a , b[then f is increasing on the interval] a , b [

2- If f ' (x) < 0 for all the values of x ∈] a , b [then f is increasing on the interval] a , b [

Example — Determining the increasing and decreasing intervals

1 Determine the increasing and decreasing intervals of the function f where $f(x) = x^3 - 3x + 2$

Solution

$\because f(x) = x^3 - 3x + 2$ is a continuous and differentiable function on $\mathbb{R}$

$\therefore f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

By putting $f'(x) = 0$ then: $3(x^2 - 1) = 3(x - 1)(x + 1) = 0$

$\therefore f'(x) = 0$ when $x = -1$ and $x = 1$

we investigate the sign of f' (x) in each of these intervals as in the opposite table of changes we find that :

f is increasing on the interval $]-\infty, -1[$

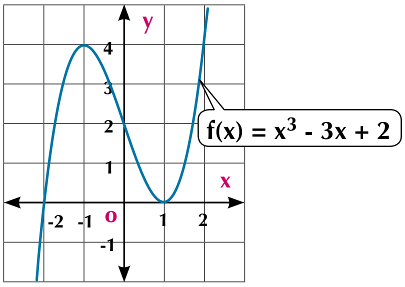
f is decreasing on the interval $]-1, 1[$

f is increasing on the interval $]1, \infty[$

x	$-\infty$	-1	1	∞
sign f' (x)	+	0 −	0 +	
behavior of f (x)	↗	↘	↗	

Note that:

1) When we sketch the curve of the function using a graphical program (opposite figure) we find that the behavior of the function curve is congruent to what we inferred in the table of changes.



2) The tangent to the curve makes an acute angle with the positive direction of x-axis in the increasing intervals and an obtuse angle with the positive direction of x-axis in the decreasing intervals.

3) The values x which separate the increasing and decreasing intervals of the function are the values at which the first derivative of the function equals zero or is not existed

Try to solve

1 Determine the increasing and decreasing intervals for each of the following :

a $f(x) = x^3 - 9x^2 + 15x$ **b** $g(x) = \frac{x}{x^2 + 1}$

Example — Trigonometric functions

2 Determine the increasing and decreasing intervals for the functions f where $f(x) = x + 2\sin x$, $0 < x < 2\pi$

Solution

f is continuous and differentiable on $]0, 2\pi[$

$\therefore f'(x) = 1 + 2\cos x$

investigating the sign of f' (x)

when $1 + 2\cos x = 0$ $\therefore \cos x = -\frac{1}{2}$

$\because x \in]0, 2\pi[$ $\therefore x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$

x	$-\infty$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	∞
sign of f' (x)	+	0 −	0 +	
behavior of f (x)	↗	↘	↗	

Note that:

when $x = \frac{\pi}{2}$ $\quad f'(x) = 1 > 0$ $\quad \therefore$ f is increasing on $]0, \frac{2\pi}{3}[$

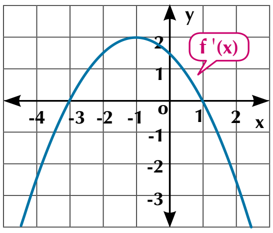
when $x = \pi$ $\quad f'(x) = -1 < 0$ $\quad \therefore$ f is decreasing on $]\frac{2\pi}{3}, \frac{4\pi}{3}[$

when $x = \frac{3\pi}{2}$ $\quad f'(x) = 1 > 0$ $\quad \therefore$ f is increasing on $]\frac{4\pi}{3}, 2\pi[$

Try to solve

2) Determine the increasing and decreasing intervals of the function f where $f(x) = x - 2\cos x$, $0 < x < 2\pi$

Critical thinking: The opposite figure shows the curve of $f'(x)$ for the function f where f (x) is polynomial.



a) Determine the increasing and decreasing intervals of the function f

b) Find the solution set of the inequality $f''(x) > 0$

Example

3) Determine the increasing and decreasing intervals of the function h where $h(x) = 2\ln x - x^2$

Solution

h (x) is differentiable for each $x \in \mathbb{R}^+$

$$h'(x) = \frac{2}{x} - 2x = \frac{2(1 - x^2)}{x}$$

x	0	1	∞
sign f' (x)	+	0	−
behavior of f (x)	↗		↘

investigating the sign of h' (x)

when $h'(x) = 0$ $\quad \therefore x = 1$ or $x = -1 \notin \mathbb{R}^+$

when $x < 1$ $\quad \therefore h'(x) > 0$ and h is increasing on $]0, 1[$

when $x > 1$ $\quad \therefore h'(x) < 0$ and h is decreasing on $]1, \infty[$

Try to solve

3) Determine the increasing and decreasing intervals of the function f where $f(x) = x - e^x$ using the geogebra program sketch the curve of the function f and check your answer.

Exercises 3 - 1

Determine the increasing and decreasing intervals of the function f for each of the following:

1. $f(x) = x^2 - 4x$
2. $f(x) = (x-3)^2$
3. $f(x) = x^3 - 6x^2 + 5$
4. $f(x) = 9x - x^3$
5. $f(x) = x^4 + 4x$
6. $f(x) = 2 - 3(x-2)^{\frac{4}{3}}$
7. $f(x) = 1 - \frac{1}{x}$
8. $f(x) = \frac{x-2}{x+2}$
9. $f(x) = \frac{\sqrt{x-1}}{x}$
10. $f(x) = x + \ln x$
11. $f(x) = 3 - \ln x^2$
12. $f(x) = 5 - 2e^{2x}$

Answer the following:

13. Prove that the function f where $f(x) = \tan x - x$ is increasing on the interval $]0, \frac{\pi}{4}[$
14. Determine the increasing and decreasing intervals of the function f where:

    $f(x) = 1 - \sin x, 0 < x < 2\pi$

15. If f and h are two differentiable functions and $f'(x) < h'(x)$ for each $x \in \mathbb{R}$, prove that the function z where $z(x) = f(x) - h(x)$ is decreasing for each $x \in \mathbb{R}$.

3 - 2 Maxima and Minima (Extrema)

You will learn

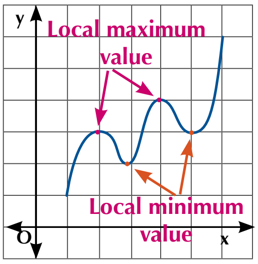
- The concept of the critical point
- The concept the local maximum and minimum of a function
- First derivative test for the local maximum and minimum values
- Finding the maximum values of a function on a closed interval

Key terms

- Critical point
- Local Maximum
- Local Minimum
- Absolute Extrema

Materials

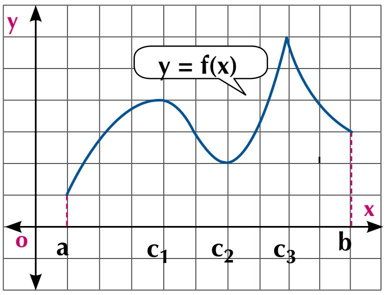
- Scientific calculator
- Computer graphics



Think and discuss

The opposite figure shows the curve of the continuous function f on [a , b]

1- Determine the increasing and decreasing intervals of the function f



2- What is the value of f' (c_1) when $x = c_1$? Describe the change of f on the interval] a , c_2 [. Is f (c_1) the greatest value of f in this interval?

3- What is the value of f' (c_2) when $x = c_2$? Describe the change of f on the interval] c_1, c_3[. Is f(c_2) the minimum value of f in this interval ?

4- Can you find the value of f' (c_3)? Explain.
Describe the change of f on the interval]c_2, b [. Is f (c_3) the maximum value of f in this interval?

Definition

The Critical Point

The continuous function f on the interval] a , b [has a critical point (c , f (c))
If c ∈] a , b [and f' (c) = 0 or the function f is not differentiable when x = c.

In the previous figure, we deduce that:
there is a critical point when $x = c_1$ and $x = c_2$ since $f'(c_1) = f'(c_2) = 0$. It is sometimes called stationary point. there is also another critical point when $x = c_3$ since f is continuous when $x = c_3$ and is not differentiable (the right derivative ≠ the left derivative).

Definition

Local Maximum and Minimum values

If f is a continuous function whose domain is I and c ∈ I, then the function f has:

a local maximum value when x = c if an open interval is found] a , b [⊂ I containing c where $f(x) \leqslant f(c)$ for each x ∈] a , b [

a local minimum value when x = c if an open interval is found] a , b [⊂ I containing c where $f(x) \geqslant f(c)$ for each x ∈] a , b [

Note that :

In think and discuss: there are local maximum values when $x = c_1$ and $x = c_3$ whereas there is a local minimum value when $x = c_2$

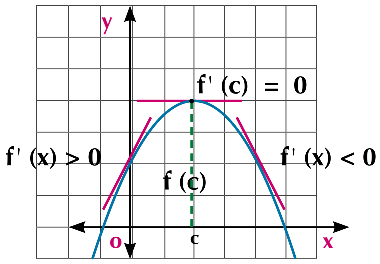
First Derivative Test of the local maximum and minimum values

Learn

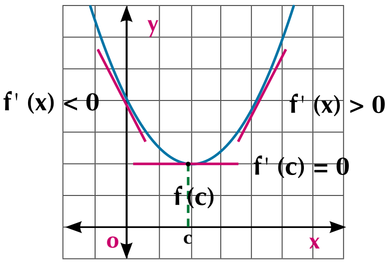
If (c, f(c)) is a critical point of the continuous function f at c and an open interval is found around c where:

1- $f'(x) > 0$ when $x < c$, $f'(x) < 0$ when $x > c$, then f (c) is a local maximum value

2- $f'(x) < 0$ when $x < c$, $f'(x) > 0$ when $x > c$, then f (c) is a local minimum value



f (c) is a local maximum value at c



f (c) is a local minimum value at c

3- If the sign of f' (x) on the two sides of c doesnot change, then the function f doesont have local maximum or local minimum values at c.

Theorem

If f is a differentiable function on]a , b[and the function has a local maximum value or local minimum value at $c \in$]a , b[
then $f'(c) = 0$ or $f'(c)$ is not existed.

First derivative test

Example

(1) If $f(x) = x^3 + 3x^2 - 9x - 7$, find the local maximum values or the local minimum values of the function f

Solution

1) Determining the critical points : f is continuous and differentiable

$\therefore f'(x) = 3x^2 + 6x - 9$

$= 3(x^2 + 2x - 3) = 3(x + 3)(x - 1)$

when $f'(x) = 0$

$\therefore x = -3$ or $x = 1$

we have two critical points

$(-3, f(-3)), (1, f(1))$

i.e. the two points : (-3 , 20) , (1 , -12)

Determine the critical points → Investigate the sign of f'(x) → + → - local maximum
→ - → + local minimum

2) The first derivative test at each critical point and the following table of changes shows it

x	$-\infty$	-3		1	∞
sign of f' (x)	+	0	−	0	+
behavior of f (x)	↗	20	↘	-12	↗

3) In the neighborhood of $x = -3$

the sign of f' (x) changes from positive (before $x = -3$) to negative (after $x = -3$)

$\therefore f(-3) = 20$ is a local maximum value.

in the neighborhood of $x = 1$ the sign of f' (x) changes from negative (before $x = 1$) to positive (after $x = 1$)

$\therefore f(1) = -12$ is a local minimum value.

Try to solve

1 If $f(x) = \frac{1}{3}x^3 - 9x + 3$, find the local maximum and minimum values of the function f

Example

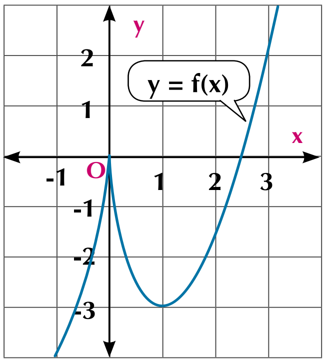
The first derivative is not existed

2 Find the local maximum and minimum values of the function f if $f(x) = x^{\frac{2}{3}}(2x - 5)$

Solution

The domain of the function f is $\mathbb{R}$ and it is continuous for each $x \in \mathbb{R}$

1) Determining the critical points:



$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(2x - 5) + 2x^{\frac{2}{3}}$$

$$= \frac{2[2x - 5 + 3x]}{3\sqrt[3]{x}} = \frac{10(x - 1)}{3\sqrt[3]{x}} \qquad x \neq 0$$

$\because$ f is continuous when $x = 0$ and f' (0) is not existed

$\therefore$ One critical point is found which is (0 , f (0)) i.e. (0 , 0)

when $f'(x) = 0$ $\therefore x = 1$ and therefore,

there is a critical point which is

(1 , f(1)) i.e. (1 , -3) as shown in the opposite figure .

2) The first derivative test at each critical point and the following table of changes shows it.

x	$-\infty$	-1		1	∞
sign of f' (x)	+	not existed	−	0	+
behavior of f (x)	↗	-	↘	-3	↗

3) When $x = 0$

there is a local maximum value $= 0$

when $x = 1$ there is a local minimum value $= -3$

Try to solve

2 Prove that the function f where $f(x) = \sqrt[3]{x^2}$ has a local minimum value .

Critical thinking : Does the function f where $f(x) = x^3 + 3x - 4$ have local maximum and minimum values? Explain.

Example Fractional functions

3 Find the local maximum and minimum values of the function f where $f(x) = x + \frac{4}{x}$ and show its type.

Solution

The domain of f = $\mathbb{R} - \{0\}$

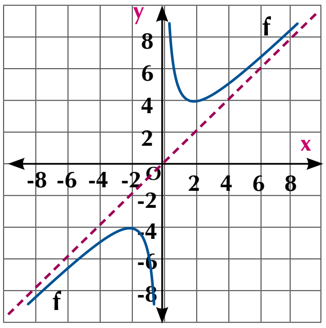
1) Determining the critical points: $f'(x) = 1 - 4x^{-2} = \frac{x^2 - 4}{x^2}$ the function has two critical points which are (2, f(2)) , (-2, f (-2)) i.e. (2 , 4) , (-2 , -4).

2) The first derivative test at each critical point which the opposite table of the function changes shows (note that x = 0 is excluded from the domain of f).

x	-∞	-2		0		2	∞
sign of f' (x)	+		−		−		+
behavior of f (x)	↗	-4	↘		↘	4	↗

3) When x = -2 there is a local maximum value = -4

when x = 2 there is a local minimum value = 4



Note that: the local maximum value of the function may be less than the local minimum value

Technology: the opposite figure shows the curve of the function f. Use a graphic program to compare between the table of the function changes and its curve? what do you notice?

Try to solve

3 Find the local maximum and minimum values of the function f where $f(x) = \frac{x^2}{1 - x}$ and show its type

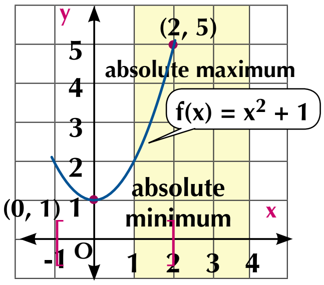
Learn

The Absalute maximum values of a Function on a Closed Interval

Definition of maximum values: if f function is defined on the closed interval [a , b] and c ∈ [a , b]

1) f (c) is the minimum value on the interval [a ,b] when $f(c) \leqslant f(x)$ for each x ∈ [a , b]

2) f (c) is a maximum value on the interval [a , b] when $f(c) \geqslant f(x)$ for each x ∈ [a , b]



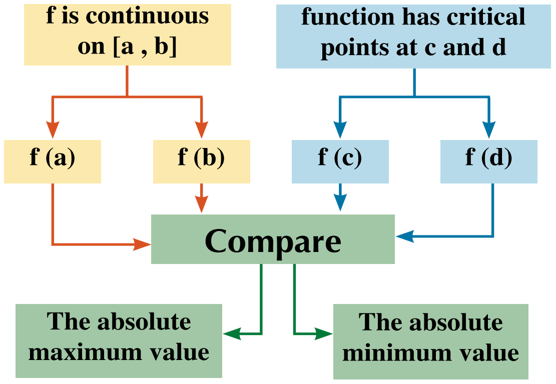
- The minimum value and maximum value of a function on an interval are called the extrema of thes function on this interval.
- The maximum value can be occured at an interior point of the interval or at the boundries of the function once it occurs at the boundries of the interval , it is called the end point extrema

Theorem

If function f is continuous on the interval [a , b], then the function f has an absolute maximum value and absolute minimum value on the interval [a , b].

To find the absolute maximum values of the function f on the closed interval [a , b] we follow up the opposite diagram as follows:

- Calculate f (a) , f (b) , and the value of the function at each critical point.
- Compare between the values above; the maximum value is the absolute maximum value and the minimum value is the absolute minimum value.



Example

4 Find the absolute maximum values of the function f where f (x) = $x^3 - 12x + 12$, $x \in [-3 , 3]$

Solution

$\because$ f (x) = $x^3 - 12x + 12$, $x \in [-3 , 3]$

$\therefore$ f (-3) = $(-3)^3 - 12(-3) + 12 = 21$ **(1)**

, f (3) = $(3)^3 - 12(3) + 12 = 3$ **(2)**

$f'(x) = 3x^2 - 12 = 3(x - 2)(x - 2)$

to determine the critical point, put $f'(x) = 0$

$\therefore$ $x = 2 \in [-3 , 3]$ or $x = -2 \in [-3 , 3]$

when x = 2 there is a critical point and: f (2) = -4 **(3)**

when x = -2 there is a critical point and: f (-2) = 28 **(4)**

By comparing the values of 1 , 2 , 3 and 4 we find that:

The function f has an absolute maximum value = 28 and an absolute minimum value = - 4

Try to solve

4 Find the absolute extrema values of the function f

a $f(x) = 10x\, e^{-x}$ $x \in [0 , 4]$

b $f(x) = \frac{4x}{x^2 + 1}$, $x \in [-1 , 3]$

Note that:

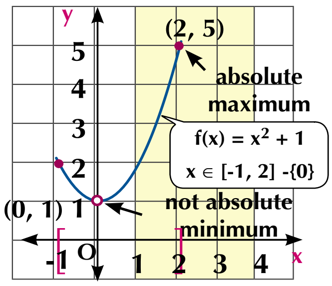


fig (1)

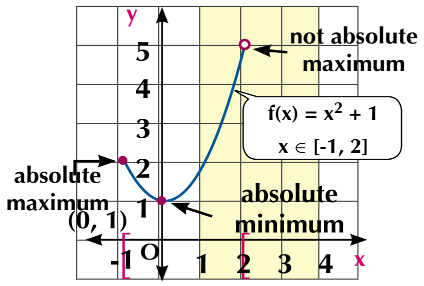
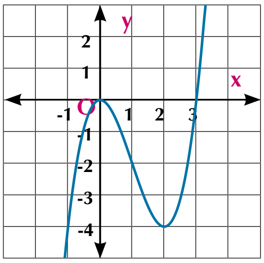
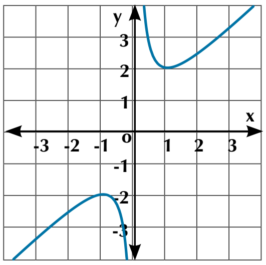


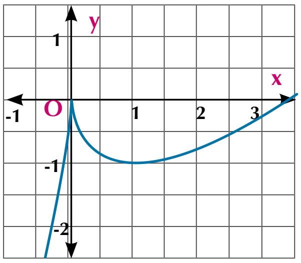
fig (2)

Exercises 3 - 2

Determine the local maximum and minimum values (if existed) of the function f in the following figures and show their types:

1 

2 

3 

Find the local maximum and minimum values (if existed) of the function f for each of the following and show their types:

4. $f(x) = x^3 + 3x^2 + 2$
5. $f(x) = x^4 - 2x^2$
6. $f(x) = 4x - x^3$
7. $f(x) = 3x^5 - 5x^3$
8. $f(x) = 3 - x^{\frac{2}{3}}$
9. $f(x) = (x + 2)^{\frac{2}{3}}$
10. $f(x) = x + \frac{4}{x^2}$
11. $f(x) = x + \frac{4}{x - 1}$
12. $f(x) = \frac{3}{x - 2}$
13. $f(x) = 4e^{-x^2}$
14. $f(x) = e^x(3 - x)$
15. $f(x) = e^x + e^{-x}$
16. $f(x) = x - \ln x$
17. $f(x) = 8\ln x - x^2$
18. $f(x) = (x - 1)\ln x$

Find the absolute maximum values of the function f on the given interval:

19. $f(x) = x^3 - 3x + 1$, $x \in [-2 , 1]$
20. $f(x) = \sqrt{x - 1}$, $x \in [2 , 5]$
21. $f(x) = \sin x + \cos x$, $x \in [0 , 2\pi]$
22. $f(x) = x e^{-x}$, $x \in [0 , 2]$

Answer the following:

23. **Critical thinking:** Find the value of each for a , b , c and d where the curve of $f(x) = ax^3 + bx^2 + cx + d$ satisfies the following conditions together:

   a. It passes through the origin point.

   b. It has a critical point when $x = 1$

   c. The equation of the tangent to the curve at point $(2 , f(2))$ on it is $9x + y = 20$.

3 - 3 Curve Sketching

You will learn

- Determining the convex intervals of the curve of the function upward and downward.
- Finding the inflection points to the curve of the function,
- Using the second derivative test to find the local maximum and minimum values
- Curve sketching

Key terms

- Convexity
- Convex upward
- Convex downward
- Inflection point

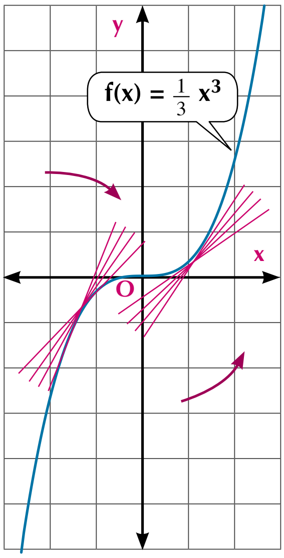
Materials

- Scientific calculator.
- Computer graphics

Explore

The opposite figure shows the curve of the function f where:

$f(x) = \frac{1}{3} x^3 , x \in \mathbb{R}$



Notice that the function f is increasing on $\mathbb{R}$. why?

Does the direction of the curve convexity in the interval $]-\infty , 0[$ differ from the direction of its convexity in the interval $]0 , \infty[$?

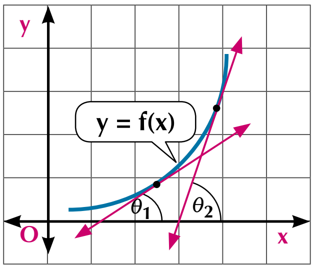
what is the location of the curve of the function in the interval $]-\infty , 0[$ with respect to all its tangents? Does the slope of the tangent f' (x) increase or decrease by the increase of the value of x?

what is the location of the curve of the function in the interval $]0 , \infty[$ with respect to all its tangent? Does the slope of the tangent f' (x) increase or decrease by the increase of the value of x?

Convexity of a curves

Definition

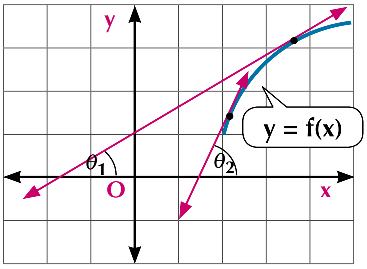
Let f be a differentiable function on the interval] a , b [, the curve of the function f is convex downward If f' is increasing on this interval and is convex upward if f' is decreasing on this interval.



The curve is convex downward

f' is increasing and its derivative is positive

i.e. $f''(x) > 0$



The curve is convex upward

f' is decreasing and its derivative is negative

i.e $f''(x) < 0$

If the function f has a non-zero second derivative, it can be used to study the increasing and decreasing of the first derivative f' and to identify the intervals of convex upwards and convex downward to the curve of the function f.

The Second Derivative Test for Convexity of curves

Theorem

Let f be a differentiable function twice on the interval]a , b[

1- If $f''(x) > 0$ for all the values of $x \in]a, b[$, then the curve of f is convex downwards on the interval]a , b[

2- If $f''(x) < 0$ for all the values of $x \in]a, b[$, then the curve of f is convex upwards on the interval]a , b[

Example Determining the intervals of polynomial convexity

1 If $f(x) = 2 - 3x^2 - x^3$, determine the intervals at which the curve of the function f is convex upwards and the intervals it is convex downwards.

Solution

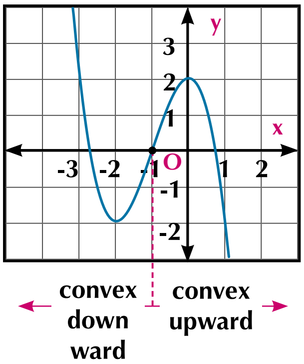
f is continuous and differentiable for each $x \in \mathbb{R}$ where:

$f'(x) = -6x - 3x^2$, $f''(x) = -6 - 6x = -6(1 + x)$

when $f''(x) = 0$ $\therefore x = -1$

The intervals of convexity: the opposite table shows the sign of f''and the intervals of the convexity of the curve of the function f upwards and downwards, **i.e:** The curve of the function is convex downwards in the interval]-∞ , -1[and convex upwards in the interval]-1 , ∞[

x	-∞	-1	∞
sign of f''	+	0	−
convexity of curve of f			



Try to solve

1 Determine the intervals of convexity upward and downward for each of the following curves:

a $f(x) = x^2 - 4x + 2$ b $g(x) = x^4 - 4x^3$

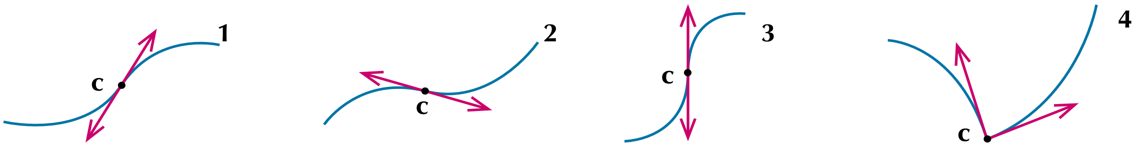
Technology: use a graphic program to sketch the curve of the two function f , h where $h(x) = \sqrt[3]{x}$, $f(x) = x^{\frac{2}{3}}$, then determine the intervals of the convexity upwards and downwards and check your answer using the second derivative test .

Notice that: The direction of the convexity of the curve of the continuous function may change from upwards to downwards or vice versa at a point the second derivative of the function gets vanished or is not existed .

Definition

The inflection point

If f is a continuous function on the open interval]a , b[, c ∈] a , b[and the curve of the function has a tangent at point (c, f (c)), then this point is called the inflection point to the curve of the function f if the convexity of the curve of the function changes at this point from being convex downward to convex upwards or vice versa.



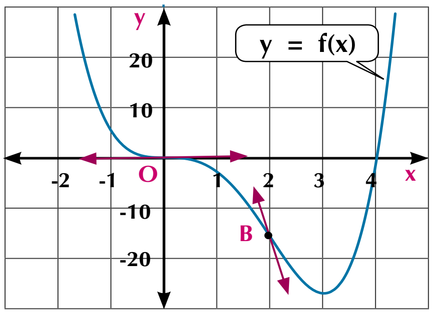
There are inflection points to the curves 1 , 2 and 3 to change the direction of the curve convexity and the existance of a tangent to it at C

There arenot inflection points due to the absence of a tangent at C

Notice that:

1- The tangent at the inflection point intersects the curve of the function since the curve at one side of this point lies under the tangent and lies on the tangent in the other side.

2- **In the opposite figure** the curve of the function f has two inflection points at origin point O (0, 0) and the other at point B (2, f(2)).



Example

Convexity and inflection point

(2) If $f(x) = \begin{cases} x^2 - 4 & \text{when } x < -2 \\ x^3 - 3x + 2 & \text{when } x \geqslant -2 \end{cases}$

Determine the convexity intervals of the curve of f upwards and downwards and find the inflection points and the equation of the tangent to the function if existed .

Solution

The function f is piece -wise function, its domain is R and is continuous when x = -2 since $f(-2)^- = f(-2)^+ = f(-2) = 0$

$$f'(-2^-) = \lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h \to 0^-} \frac{(-2+h)^2 - 4 - 0}{h} = \lim_{h \to 0^-} (h - 4) = -4$$

$$f'(-2^+) = \lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h}$$

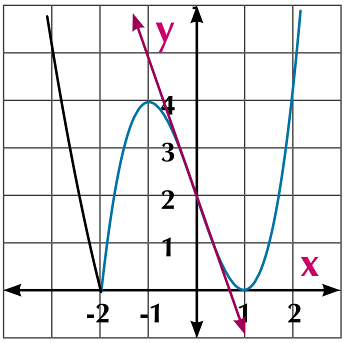
$$= \lim_{h \to o^+} \frac{(-2+h)^3 - 3(-2+h) + 2 - 0}{h} = \lim_{h \to o^+} (h^2 - 6h + 9) = 9$$

$\because f'(-2^-) \neq f'(-2^+)$ $\quad \therefore$ The function is not differentiable when x = -2

$$f'(x) = \begin{cases} 2x & \text{when } x < -2 \\ 3x^2 - 3 & \text{when } x > -2 \\ \text{is not existed when } x = -2 & [f'(-2^-) \neq f'(-2^+)] \end{cases}$$

$$f''(x) = \begin{cases} 2 & \text{when } x < -2 \\ 6x & \text{when } x > -2 \end{cases}$$

The following table shows the sign of f'' and the intervals of convexity of the curve upwards and downward.



x	$-\infty$	-2		0		∞
Sign of f''	+	**is not existed**	-		+	
Convexity of curve of f	⌣		⌢		⌣	

Intervals of convexity: the curve of f is convex downwards in the interval $]-\infty, -2[$ and the interval $]0, \infty[$ and it is convex upwards in the interval $]-2, 0[$

Inflection points

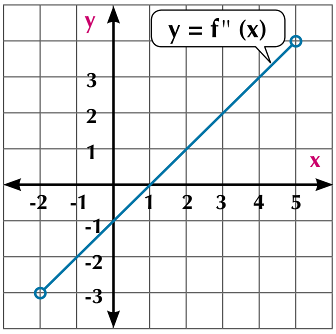
➲ Point (-2 , f(-2)) i.e. (-2 , 0) is not an inflection point to the curve of f although the changes of the direction of its convexity is changes around it due to the absence of the tangent to the curve of the function at this point (f'(x) is not found)

➲ Point (0 , f(0)) i.e (0, 2) is the inflection point to the curve of f due to the change of its convexity around it . The tangent to the curve of the function is existed at this point and intersects it at this point, its slope $f'(x) = -3$ and its equation is: $y - 2 = -3x$ (as in the graph)

Try to solve

(2) If $f(x) = \begin{cases} (x+3)^2 & \text{when } x < -1 \\ 3x^2 - x^3 & \text{when } x \geqslant -1 \end{cases}$

Determine the intervals of the convexity upwards and downwards to the curve of the function f and find the inflection points and the equation of tangent of the curve at it.



Critical thinking : The opposite figure represents the curve of $f''(x)$ on the interval $]-2, 5[$ of the continuous function f.

➲ Find the convexity of the intervals upwards and downwards to the curve of the function f if found .

➲ Are there inflection points to the curve of function f in this interval? Explain.

Second derivative test of the local maximum or minmum values

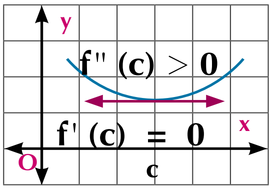
Theorem

Let f be a differentiable function twice on an open interval containing c where $f'(c) = 0$

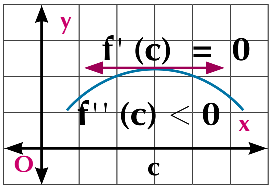
1) If $f''(c) < 0$ then f(c) is a local maximum value.

2) If $f''(c) > 0$ then f(c) is a local minimum value.

3) If $f''(c) = 0$ then the second derivative test cannot determine the type of point(c, f(c)) whether it is local maximum or minimum.



f(c) local minimum value



f(c) local maximum value

Example

3 Use the second derivative test to find the local maximum and minimum values of the function f where : $f(x) = x^4 - 8x^2 + 10$

Solution

f(x) is polynomial, so it is continuous and its domain is $\mathbb{R}$

$f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$, $f''(x) = 12x^2 - 16$

The function has critical points when $f'(x) = 4x(x^2 - 4) = 0$ i.e when x = 0, x = 2, x = -2

The second derivative test for existance of the local maximum and minimum values:

when x = 0 $f''(0) = -16 < 0$ $\therefore$ f(0) = 10 local maximum value

when x = 2 $f''(2) = 32 > 0$ $\therefore$ f(2) = -6 local minimum value

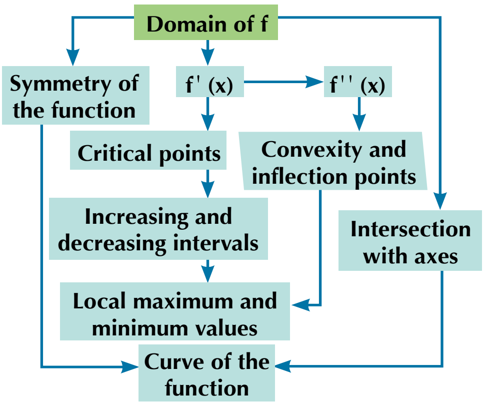
when x = -2 $f''(-2) = 32 > 0$ $\therefore$ f(-2) = -6 local minimum value

Try to solve

3 Use the second derivative test to find the local maximum and minimum values of function f where $f(x) = x^3 - 3x^2 - 9x$ and check your answer using the graphical calculator or graphic programs.

Curve Sketching for Polynomials

Differentiation is used for sketching the curves of the functions. It depends on tracing the behavior of f(x) of function f when the value of x changes at a certain interval and representing the ordered pairs (x , y) in the perpendicular coordinate plane where y = f(x). In our study, we will focus on sketching the curves of the functions on the polynomial functions of the third degree and less in the form of $f(x) = a x^3 + b x^2 + c x + d$ to sketch the curve of function f where y = f(x), follow the opposite diagram as follows :



1- If the function is even, its curve is symmetrical about y-axis and symmetrical about the origin point if f is odd.

2- Studying the changes of the function and determining the convexity intervals and the inflection points if found and the local maximum and minimum values if found.

3- Doing the increasing , decreasing and convexity table in order to know the curve sketching and the critical points.

4- Finding the intersection points of the curve of the function with the two axes of coordinates if possible.

5- Sketching the curve of the function and you can use some additional points to improve your sketching.

Example

Sketching the curve of the function

(4) Sketch the curve of the function f where $y = f(x) = x^3 - 3x^2 + 4$

Solution

1- The function f is polynomial , its domain is R , and it is neither even nor odd.

2- $f'(x) = 3x^2 - 6x = 3x(x - 2)$, $f''(x) = 6x - 6 = 6(x - 1)$

the function has critical points at $f'(x) = 0$ i.e when $x = 0$, $x = 2$

the function is increasing at the interval $]-\infty, 0[$ and at the interval $]2, \infty[$ and it is decreasing at the interval $]0, 2[$

$f''(x) = 0$ when $x = 1$

$f''(x) < 0$ at the interval $]-\infty, 1[$ so the curve is convex upwards at this interval,

$f''(x) > 0$ at the interval $]1, \infty[$ so the curve is convex downwards at this interval

Point (1 , f(1)) i.e. (1 , 2) is an inflection point.

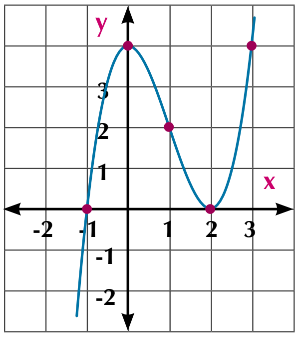
3- Increasing , decreasing and convexity table

x	$-\infty$		0		1		2		∞
Sign of f'		+	0		-		0	+	
Behavior of f		↗			↘			↗	
Sign of f''			-		0		+		
Convexity of f		∩					∪		
y			4		2		0		
			Local maximum value		**Inflection point**		**Local minimum value**		

4- Intersection points with coordinate axes: (0, 4) , (2, 0)

5- Curve sketching of the function f

Additional points : (-1 , f(-1)) i.e. (-1 , 0) (3 , f(3)) i.e. (3 , 4)



Try to solve

(4) Sketch the curve of function f where $y = f(x) = 12x - x^3$

Example

Curve sketching of the function

(5) Sketch the curve of the function f where y = f(x) if you know the following:

1- f is a continuous function whose domain is [1 , 7] , f(1) = -2 , f(5) = 4

2- $f'(5) = 0$, $f'(x) > 0$ when $x < 5$, $f'(x) < 0$ when $x > 5$

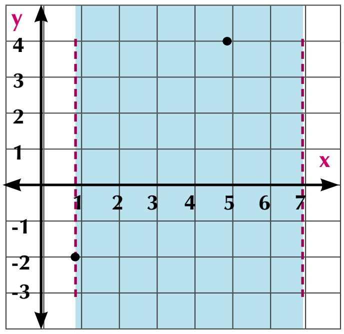
3- $f''(x) < 0$ when $1 < x < 7$

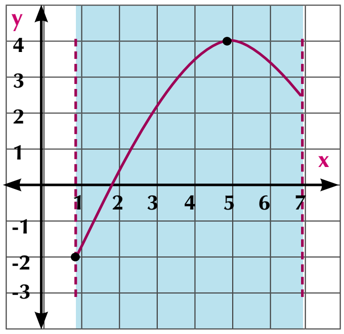
Solution

From (1): we graph the two perpendicular coordinate axes.
the two points (1, -2) and (5, 4) in the domain [1, 7].

from (2): when x = 5 the tangent // x-axis and f is increasing on the interval] 1, 5[and decreasing on the interval] 5, 7[

from (3): the curve is convex upwards on] 1, 7[





Try to solve

(5) Sketch the curve of the function f where y = f(x) if you know the following :

1- f is continuous whose domain is [0, ∞[, f(4) = 3 , f(0) = 1

2- f' (x) > 0 when x > 0

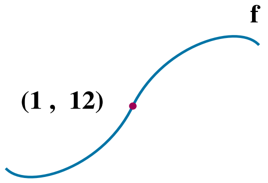
3- f'' (x) > 0 when x < 4 , f'' (4) = 0 , f''(x) < 0 when x > 4

Example

Solving the equations

(6) If point (1, 12) is the inflection point to the curve of the function f where $f(x) = ax^3 + b\,x^2$, find the real values of a and b.

Solution



∵ Point (1 , 12) is an inflection point to the curve of f

∴ f'' (1) = 0 **(1)** , f(1) = 12 **(2)**

$f'(x) = 3\,a\,x^2 + 2\,b\,x$, $f''(x) = 6\,a\,x + 2\,b$

from (1): 6 a + 2 b = 0 ∴ b = - 3 a

from (2): a + b = 12 ∴ a - 3 a = 12 and a = -6 , b = 18

Try to solve

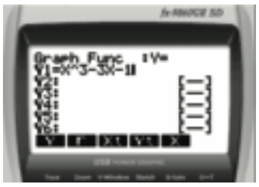
(6) If point (2, 2) is the inflection point to the curve of the function f where $f(x) = x^3 + a\,x^2 + b\,x$, find the real values of a and b.

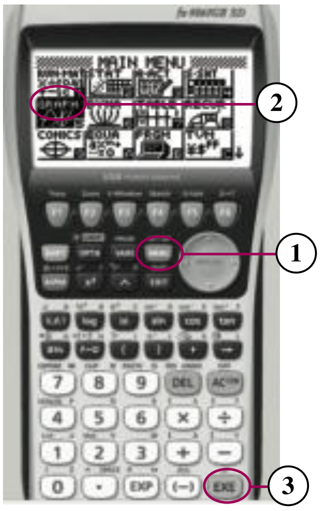
Activity

Using the graphical calculator to sketch the functions.

To use the graphical function to sketch the curve of the function f where $f(x) = x^3 - 3x + 1$ follow up the next steps:

1- Turn on the calculator and press MENU then move the arrows on the screen and select GRAPH, and press EXE the entry button to get the typing window .

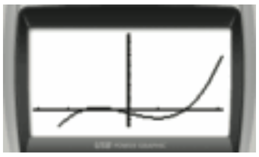




2- Write down at Y1 in the typing window the function you want to graph where a button is used to type the variable x so press the next buttons :

Start → [T, θ, X] [^] [3] [−] [3] [T, θ, X] [+] [1]

3- To graph the function press **start** → [EXE] [▲] [EXE] and the graphing window appears as shown in the opposite figure.

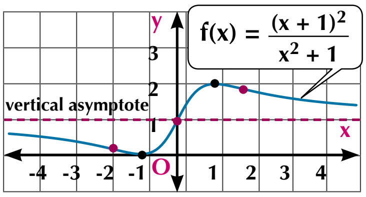


4- Use the button [◁▷] in the graphing window to study the behavior of the function and to identify the convexity upwards and downwards.

Technology: it is difficult to graph the curves of some functions. You can use geogebra program or any other graphic program to graph the curve of the function and to study its properties.

The opposite figure shows the curve of the function f where

$$f(x) = \frac{(x+1)^2}{x^2+1}$$



Notice:

1) The critical points: the curve has critical points when $x = -1$, $x = 1$

when $x = -1$ $f(-1) = 0$ local minimum value, when $x = 1$ $f(1) = 2$ local maximum value .

2) Convexity intervals: upwards: $]-\infty, -\sqrt{3}[$, $]0, \sqrt{3}[$, and downwards: $]-\sqrt{3}, 0[$, $]\sqrt{3}, \infty[$

3) Inflection point: when $x = -\sqrt{3}$, there is a tangent intersects the curve of f and when $x = \sqrt{3}$, there is a tangent intersects the curve of f

4) Curve sketching: the two ends to the curve of the function approach the straight line $y = 1$ and it is called the vertical asymptote to the curve of the function and its equation is $y = a$

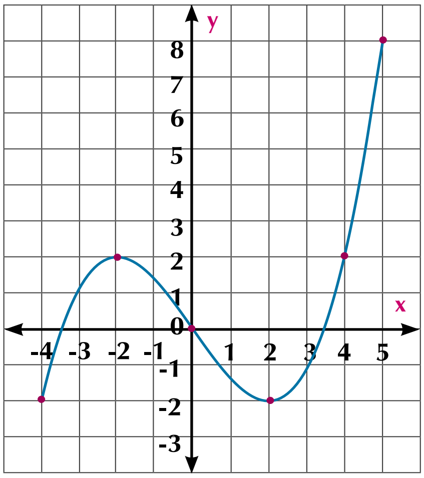
where: $a = \lim_{|x| \to \infty} f(x) = \lim_{|x| \to \infty} \frac{(x+1)^2}{x^2+1} = 1$

Application: Graph the two curves of the two functions using a graph program then study the properties for each of them:

$f(x) = \frac{4x^2}{x^2+3}$ $\qquad g(x) = \frac{x^2 - 4x}{x^2 - 4x + 3}$

Exercises 3 - 3

1 **The opposite figure shows the curve of the function f where y = f(x), complete:**



a) The domain of f = ...

b) f' (x) = 0 when x ∈ ....

c) f'' (x) > 0 when x ∈ ....

d) The curve is convex upwards when x ∈ ....

e) The inflection point of the curve is ....

f) The function has a local minimum value when x = ...

g) The function has absolute maximum value equals ....

Investigate the convexity intervals of the function f, find the coordinates of the inflection points (if found) for each of the following:

2 $f(x) = 4 - 6x - 3x^2$

3 $f(x) = x^3 - 3x^2 + 1$

4 $f(x) = 15x + 6x^2 - x^3$

5 $f(x) = x^4 - 8x^2 + 16$

6 $f(x) = \frac{6}{x^2 + 3}$

7 $f(x) = \frac{x^2 - 1}{x^2 - 4}$

8 $f(x) = 3$

9 $f(x) = \begin{cases} x^3 - 3x & \text{when } x < 0 \\ 4x - x^2 & \text{when } x \geqslant 0 \end{cases}$

10 Prove that the measure of the angle of inclination of the tangent at the inflection point to the curve of the function f where $f(x) = \frac{x}{1 - x^2}$ equals $\frac{\pi}{4}$

11 If the curve of the function f where $f(x) = x(x - 3)^2$ has a local maximum value at x_1 and a local minimum value at x_2, prove that x-coordinate of the inflection point $= \frac{x_1 + x_2}{2}$

12 Find a , b where the curve $x^2 y + a y + b x^2 = 0$ has an inflection point at point (1 , -1).

Sketch the curve of the continuous function f which has the given properties for each of the following:

13 f(0) = 4 f(3) = 4 , $f'(x) < 0$ for each $x < 2$, $f'(x) > 0$ for each $x > 2$, $f''(x) > 0$

14 f(1) = f(5) = 0 , $f'(x) < 0$ for each $x < 3$, $f'(x) > 0$ for each $x > 3$, $f''(x) < 0$ for each $x \neq 3$

15. $f(-1) = 2$ $f(0) = 4$, $f(1) = 0$, $f'(1) = f'(-1) = 1$, $f''(x) < 0$ for each $x < 0$, $f''(x) > 0$ for each $x > 0$

16. $f(3) = 4$, when $x < 3$ then $f'(x) > 0$, $f''(x) > 0$ when $x > 3$ then $f'(x) < 0$, $f''(x) > 0$

Study the changes of the function f and sketch its curve for each of the following:

17. $f(x) = x^2 - 6x + 5$

18. $f(x) = 3 - x^2$

19. $f(x) = x^3 - 3x^2 + 3$

20. $f(x) = \frac{1}{3}x^3 - x + 2$

21. $f(x) = \frac{1}{8}x^3 - \frac{3}{2}x + 1$

22. $f(x) = -x(x-3)^2$

23. $f(x) = (2-x)(x+1)^2$

24. $f(x) = \frac{1}{8}(x+4)(x-2)^2$

25. $f(x) = \begin{cases} x^3 - 3x^2 & \text{when } x > 0 \\ x^2 - 2x & \text{when } x \leqslant 0 \end{cases}$

26. $f(x) = x|x-4|$

3 - 4 Applications of Maxima and Minima

You will learn

- Mathematical Modeling

Key terms

- Mathematical Modeling

Materials

- Scientific calculator
- Computer graphics

Mathematical Modeling

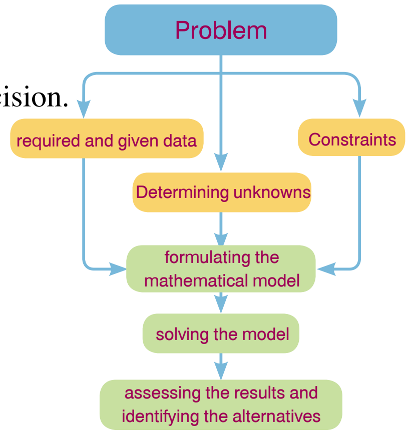
The process of taking a scientific decision to solve a problem gets through several stages as follows:

1- Determining the problem (aim and abilities).

2- Putting an ideational model or conceive the dimensions of the problem.

3- Finding a proper scientific model.

4- Solving the model and taking a decision.



mathematical modeling is to model a problem in regard to mathematical relations called the mathematical model this model **can be summarized in the opposite diagram where it includes:**

1- Determining the ends and components of the problem investigated (better profit - less cost- bigger area)

2- Determining the problem unknowns which their values should be found to reach the required end.

3- Showing the relations among unknowns (equations - inequalities).

4- Modeling a mathematical model; representing the problem mathematically to be solved.

5- Solving the mathematical model and interpreting its results in regard to the problem nature .

6- Determining the available alternatives if the problem has more than a solution.

Differentiation helps to solve the mathematical model for most of the practical life problems when the aim is to get the greatest value or least value of a variable in regard to the local maximum values and the absolute extrema value as in the next examples.

Example — First derivative test

(1) Find the two dimensions of a rectangle which has the biggest area and it can be drawn inside a triangle whose base length is 16cm and its height is 12 cm such that one of its sides lies on the base of the triangle and the two vertices of the opposite side lie on the two other sides of the triangle .

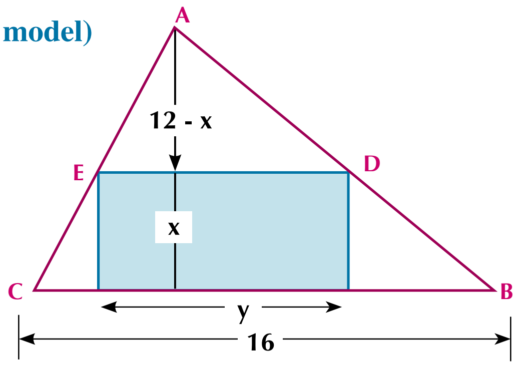
Solution

1- To calculate the biggest area, we model the problem according to the given data and constraints.

2- Determine the variables (unknowns) by letting the rectangle width = x cm, its length= y cm and area = A cm^2

3- The relations among the variables **(mathematical model)**

Area of rectangle $A = x \times y$



4- Putting the mathematical model in one variable if possible

$\frac{y}{16} = \frac{AD}{AB} = \frac{12 - x}{12}$ **(from similarity)**

$\therefore \; y = \frac{4}{3}(12 - x), \; x \in [0, 12]$ **(1)**

Area of rectangle $A = \frac{4}{3} x (12 - x)$

i.e: $A = f(x) = 16x - \frac{4}{3}x^2$ **(2)**

5- Solving the mathematical model : by differentiating both sides of relation **(2)** with respect to x.

$\therefore \; f'(x) = 16 - \frac{8}{3}x, \; f''(x) = -\frac{8}{3}$

When $f'(x) = 0$ $\quad \therefore x = \frac{16 \times 3}{8} = 6$ and at which $f''(x) < 0$

$\because$ A has an only critical point when x = 6 , and the second derivative is always negative, then this critical point gives the absolute maximum value .

$\therefore$ The function A has an absolute maximum value when $x = 6$ and $y = \frac{8}{3}(12 - 6) = 8$

i.e: the rectangle has the biggest area when its two dimensions are 6cm , 8cm

Try to solve

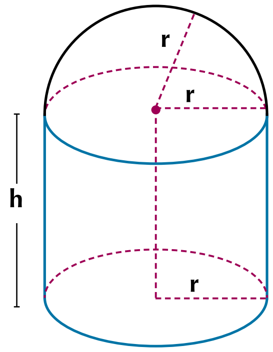
1 Find the maximum area for an isosceles triangle inscribed in a circle with radius length 12cm.

Example

Calculating the minimum cost

2 A semi - spherical top vertical cylinder-like silo is needed to be constructed where it holds $108\pi m^3$ of wheat to store **(let the wheat be stored in the cylinderic part only but not the top),** and the cost of the surface unit area from its top is twice the cost of the unit area of the lateral wall. What are the dimensions of the silo which make the cost minimum?

Solution



1- To calculate the minimum cost, graph the problem in regard to the given data and restrictions.

2- Determining the variables: let the height of the cylinder = h meter, its base radius length = r meter, the cost of the area unit of the wall = C LE, the cost of the area units of the top= 2C LE and the total cost =K L.E .

3- The relation among the variables (Modeling):

The surface area of the cylinderic surface = perimeter of the base × height = $2\pi rh$ unit area

The area of the semi -spherical surface = $\frac{1}{2}$ area of the sphere = $2\pi r^2$ unit area

the total cost $k = 2\pi rh \times c + 2\pi r^2 \times 2c = 2\pi rc(h + 2r)$

4- Putting the mathematical model in one variable:

$\because$ The volume of the cylindric part = 108π $\therefore \pi r^2 h = 108\pi$ **i.e** $h = \frac{108}{r^2}$

The total cost $k = 2\pi rc(\frac{108}{r^2} + 2r)$

$k = f(r) = 216\pi c r^{-1} + 4\pi c r^2$

5- Solving the model: $f'(r) = -216\pi c r^{-2} + 8\pi c r$

the critical point = when $f'(r) = 0$ $\therefore r^3 = \frac{216}{8}$ **i.e.** $r = 3$(unique point)

Second derivative test:

$\because f''(r) = 432\pi c r^{-3} + 8\pi c$ $\therefore f''(3) > 0$

i.e.: When the vertical cylinder radius length is 3 meters , the silo has the minimum cost and its height is $\frac{108}{9} = 12$ m.

Try to solve

2. A closed box-like tank whose capacity is 252 cubic meters and base is squared. If we want to coat the interior walls with an insulating material. The cost of the bootm is 50 LE per square meter, the cost of cover is 20 LE per square meter and the cost of the walls is 30 LE per square meter , find the dimensions of the box which makes the cost as minimum as possible.

Example

Life applications

3. A 2-meter wall and it is 2 meters away from a house, find the minimum length of the ladder that can be used to connect the ground and the house resting on the wall .

Solution

1- To calculate the minimum length of a ladder , we draw the problem in regard to the given data and restrictions .

2- Determining the variables: let the:

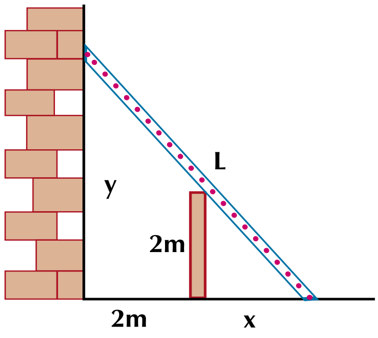
ladder length = L meter, the height of the ladder top from the ground = y meter, and the distance between the ladder base and the wall = x meter.

3- Modeling the problem:

From pythagoras : $L^2 = (x + 2)^2 + y^2$ **(1)**

from similarity: $\frac{y}{2} = \frac{x+2}{x}$

$\therefore y = \frac{2x+4}{x} = 2 + 4x^{-1}$ **(2)**



To find the minimum length of the ladder, it is enough to get L^2 the minimum value

4- Solving the model by differentiating both sides of the relation (1) and (2) with respect to x.

$\therefore \frac{d}{dx}(l^2) = 2(x+2) \times 1 + 2y\frac{dy}{dx}$, $\frac{dy}{dx} = \frac{-4}{x^2}$

$\therefore \frac{d}{dx}(l^2) = 2(x+2) + 2\left(\frac{2x+4}{x}\right) \times \frac{-4}{x^2} = 2(x+2)\left(1 - \frac{8}{x^3}\right)$

At the critical point: $\frac{d}{dx}(l^2) =$ zero

$\therefore x = -2$ is refused or $\frac{8}{x^3} = 1$

$\therefore x = 2$

x	2	
sign of $\frac{d}{dx}(\ell^2)$	−	+
ℓ^2	↘	↗

From the first derivative test of increasing and decreasing, we notice that the sign of $\frac{d}{dx}(\ell^2)$ changes from - to +

$\therefore$ When $x = 2$ then ℓ^2 is the most minimum

By substituting in (2) $\therefore y = \frac{2 \times 2 + 4}{2} = 4$

By substituting in (1)

$\therefore \ell^2 = (2+2)^2 + (4)^2 = 32$ $\therefore l = 4\sqrt{2}$

i.e.: The minimum length of the ladder connecting the ground to the house equals $4\sqrt{2}$ meters

Try to solve

3) In a perpendicular coordinate plane, $\overleftrightarrow{AB}$ is drawn to pass through point C (3 , 2) and intersect the coordinate axes at point A and point B . Prove that the minimum area of triangle AOB equals 12 squared units where O is the origin point (0, 0).

Example — Circular sector

4) A circular sector -like coin whose area is 16 cm^2 . Find the radius length of the sector circle which makes its perimeter as minimum as possible. What is the measure of its angle then?

Solution

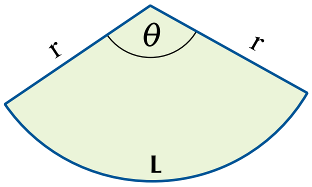
Let the arc length of the sector be l cm , and the radius length of the sector circle = r cm

$\therefore$ Perimeter of the sector $P = 2r + l$ **(1)**

$\because$ Area of the sector $= \frac{1}{2} l r = 16$ $\therefore l = \frac{32}{r}$

By substituting in (1) $\therefore P = 2r + \frac{32}{r}$ **(2)**

by differentiating both sides of the relation (2) with respect to r



$\frac{dP}{dr} = 2 - \frac{32}{r^2}$, $\frac{d^2P}{dr^2} = \frac{64}{r^3}$

When $\frac{dP}{dr} = 0$ $r = 4$, $\frac{d^2P}{dr^2} > 0$

$\therefore$ When $r = 4$ the perimeter of the sector is as minimum as possible

$\because$ Area of the sector $= \frac{1}{2} r^2 \theta^{rad}$ $\therefore \theta = \frac{16 \times 2}{4 \times 4} = 2^{rad}$

Try to solve

4) If the perimeter of a circular sector = 12cm, find the angle measure of the sector which makes its area as maximum as possible.

Exercises 3 - 4

1. The sum of two numbers is 30 and their product is as maximum as possible, find the two numbers.

2. The sum of two positive integer numbers is 5 and the sum of the cube of the smaller number and double the square of the other is as minimum as possible, find the two numbers.

3. Find the positive number which if its multiplicative inverse is added to it, the sum is as minimum as possible.

4. Find the maximum area of a rectangle -like piece of land that can be surrounded by a 120-meter fence.

5. If the perimeter of a circular sector is 30 cm and its area is as maximum as possible , find the radius length of its circle.

6. A cuboid -like box whose base is in the form of square. If the sum of all its edges equals 240cm, find the dimensions of the box that will maximize its volume.

7. If the hypotenuse length of a right-angled triangle equals 10cm, find the lengths of the two legs of the right angle when the area of the triangle is as maximum as possible.

8. An open field is bounded by a straight river from one of its sides. Determine how to place a fence around the other sides of the rectangle -like piece of land to surround as maximum area as possible by a 800 -meter fence. What is the area of the land then?

9. A closed cylindaric -like boxes are made to can beverages. The capacity of each box is k of volume units with minimum amount of the material. Find the ratio of the height (h) of the box to the radius length (r) of its base

10. A rectangle - like playground ends in two semi-circles. If the perimeter of the playground is 420 m, find the maximum area of the playground.

11. A right -angled triangle whose hypotenuse length is 30cm. Find the length of the two legs of the right angle if the length of the altitude drawn from the right angle on the hypotenuse is as maximum as possible .

12. A rectangle -like piece of cardboard paper whose two dimensions are 15cm and 24 cm. Congruent squares of side length x cm are cut from its four corners, then the projected parts are folded upward to form a box without a cover, calculate the dimension of the box when its volume is as maximum as possible.

13. An open tank of squared base, vertical sides and holds a certain amount of water, prove that the cost of coating the tank from interior by a regular insulating material is as minimum as possible if its depth equals half the side length of the base.

(14) Find the nearest point to point (0, 5) and lies on the curve $y = \frac{1}{2} x^2 - 4$.

(15) Find the shortest distance between the straight line $x - 2y + 10 = 0$ and the curve $y^2 = 4x$.

(16) A B C is a triangle where a , b . are constants, find the measure of the angle included between them which makes the area of the triangle as maximum as possible .

(17) The current intensity I (Ampere) in a circuit for the alternating current at any moment t (second) is given by the relation $I = 2 \cos t + 2 \sin t$. What is the maximum value of the current in this circuit?

(18) The volume of a bacterial culture placed in a food medium grows in regard to the relation $f(n) = 2000 + \frac{5000\, t}{100 + t^2}$ where time t is measured in hours, identify the maximum value of the culture volume.

(19) A B C D is a square whose side length is 10cm and $M \in \overline{BC}$ where B M = x cm and $N \in \overline{CD}$ where $CN = \frac{3}{2} x$, find the value of x which makes the area of $\triangle AMN$ as minimum as possible.

(20) $\overline{AB}$ is a diameter in a circle of a radius length r, two tangent are drawn to the circle at A and B from point E on the circle . Another tangent is drawn to the circle to intersect the two previous tangents at D and C respectively, prove that the minimum area of the trapezium ABCD equals $2r^2$ squared unit.

Unit Summary

First derivative test for the monotony of the functions:

Theorem: let f be a differentiable function on the interval]a , b[:

1- If $f'(x) > 0$ for all the value of $x \in$]a , b [then f is increasing on]a , b[.

2- If $f'(x) < 0$ for all the values of $x \in$]a , b[then f is decreasing on]a , b[.

The critical point:

The continuous function f on]a , b[has a critical point (c , f (c)), if $c \in$]a , b[, $f'(c) = 0$ or $f'(c)$ is not existed.

The local maximum and minimum values

If f is a continuous function whose domain is I and $c \in I$, then the function f has:

1- A local maximum value at c if an open interval is found]a , b[$\subset$ I containing c where $f(x) \leqslant f(c)$ for each $x \in$]a , b[.

2- A local minimum value at c if an open interval is found] a , b [$\subset$ I containing c where $f(x) \geqslant f(c)$ for each $x \in$]a , b[.

First derivative test of the maximum and minimum values:

If (c , f (c)) is a critical point for the continuous function f at c , and an open interval is found around c where:

1- $f'(x) > 0$ when $x < c$, $f'(x) < 0$ when $x > c$, then f (c) is a local maximum value.

2- $f'(x) < 0$ when $x < c$, $f'(x) > 0$ when $x > c$, then f (c) is a local minimum value.

Theorem: If f is differentiable on]a , b[and the function f has a local maximum value or a local minimum value at $c \in$]a , b[, then $f'(c) = 0$

The extrema values of a function on a closed interval:

Theorem: If the function f is continuous on the interval [a , b], then the function f has an absolute maximum value and absolute minimum value on the interval [a , b].

Convexity of curves

Let the function f be differentiable on the interval]a , b[, the curve of the function f is convex downwards if f' is increasing on this interval and is convex upwards if f' is decreasing on this interval.

Second derivative test for convexity of curves

Theorem: Let the function f be differentiable twice on the interval]a , b[, then:

1- If $f''(x) > 0$ for all the value of $x \in$]a , b[, then the curve is convex downwards on the interval]a , b[.

2- If $f''(x) < 0$ for all the value of $x \in$]a , b[, then the curve is convex upwards on the interval]a , b[.

Unit Summary

Inflection point

If f is a continuous function on the open interval]a , b[, c ∈] a , b[and the curve of the function has a tangent at point (c, f (c)), then this point is called the inflection point to the curve of the function f, if the convexity of the curve of the function changes at this point from being convex downward to upward or vice versa.

Second derivative test of the local maximum and minimum values

Theorem: let the function f be a second derivative on an open interval containing C where $f'(c) = 0$

1- If $f''(c) < 0$ then f(c) is a local maximum value.

2- If $f''(c) > 0$ then f(c) is a local minimum value.

Curves of polynomials

To sketch the curve of a polynomial f where y = f(x), follow up the next steps:

1- If the function is even, its curve is symmetrical about y-axis and symmetrical about the origin point if f is odd.

2- Learning the changes of the function and determining the convexity intervals and the inflection points if found and the local maximum and minimum values if found.

3- Doing the increasing , decreasing and convexity table in order to know the curve sketching and the critical points.

4- Finding the intersection points of the curve of the function with the two axes of coordinates if possible.

5- Sketching the curve of the function, you can use some additional points to improve your sketching

General Exercises

Choose the correct answer

1. If f: [-2, 4] ⟶ $\mathbb{R}$, f(x) = $x^3 - 3x$, then the number of the critical points for the function f equals:
   - a) 0
   - b) 1
   - c) 2
   - d) 3
2. The function f has a local minimum value if f(x) equals:
   - a) $2x - 1$
   - b) $x^2 + 1$
   - c) x^3
   - d) $-x^3$
3. The curve of the function f is convex downwards in $\mathbb{R}$ if f(x) equals:
   - a) $3 - x^2$
   - b) $3 - x^3$
   - c) $3 - x^4$
   - d) $3 + x^4$
4. If the curve of the function f has an inflection point when x = 2 where $f(x) = x^3 + Kx^2 + 4$, then the value of K equals:
   - a) - 6
   - b) - 3
   - c) 3
   - d) 6
5. The function f has a local maximum value if f(x) equals:
   - a) $x^2 - 2$
   - b) $x^3 + 1$
   - c) $x^3 + 3x$
   - d) $x^4 - 2x^2$

Determine the increasing and decreasing intervals of the function for each of the following:

6. $f(x) = (x - 3)^2$
7. $f(x) = 2x^3 - x^2$
8. $f(x) = (x + 4)^3$
9. $f(x) = 5 + 4x - x^2$
10. $f(x) = x^2(x - 2)$
11. $f(x) = x^4 - 4x^3$
12. $f(x) = 1 + \frac{1}{x}$
13. $f(x) = \frac{x}{x + 2}$
14. $f(x) = \sqrt[3]{x - 1}$

Determine the intervals of convexity upwards, downwards and the inflection points (if found) for each of the following:

15. $f(x) = (x - 1)^2$
16. $f(x) = x^3 - 3x^2$
17. $f(x) = x^3 - 6x^2 + 9$
18. $f(x) = x^3 - 12x + 7$
19. $f(x) = 6x^3 - x^4$
20. $f(x) = x^2 - \frac{8}{x}$

Find the local extrema values of the function f if found for each of the following:

21. $f(x) = 4 - 6x - x^2$
22. $f(x) = 2x^3 - 3x^2 + 7$
23. $f(x) = x(x - 2)^2$
24. $f(x) = x^4 - 8x^2 + 8$
25. $f(x) = \frac{x^2 - 1}{x^2 + 1}$
26. $f(x) = \frac{x^2 - 3}{x - 2}$
27. $f(x) = \frac{x - 4}{x^2 + 9}$
28. $f(x) = \sqrt[3]{(x - 2)^2}$
29. $f(x) = \sqrt[3]{x^2 - 4}$

Find the absolute extrema values of the function f in the given interval for each of the following:

30. $f(x) = x^3 - 9$ [0 , 2]
31. $f(x) = x^3 - 12x + 16$ [-3 , 5]
32. $f(x) = x^4 + 2$ [-1 , 2]
33. $f(x) = 2x^4 - x^2 + 1$ [-1 , 1]

34 $f(x) = x(x^2 - 12)$ [-1 , 4]   35 $f(x) = (x - 1)(x - 2)^2$ [0 , 5]

36 $f(x) = \begin{cases} x^3 - 3x^2 & \text{when } x \leq 0 \\ x^2 - 2x & \text{when } x > 0 \end{cases}$ [-3, 3]

Sketch the curve of the continuous function f which has the given properties for each of the following:

37 $f(4) = 3$, $f'(4) = 0$, $f''(x) < 0$ for all the values of x.

38 $f(2) = 5$, $f'(2) = 0$, $f''(x) > 0$ when $x < 2$, $f''(x) < 0$ when $x > 2$

39 $f(-3) = 8$, $f(0) = 4$, $f(3) = 0$, $f'(x) > 0$ when $x < -3$ when $x > 3$, $f''(x) < 0$ when $x < 0$, $f''(x) > 0$ when $x > 0$

40 $f(0) = 3$, $f'(2) = f'(-2) = 0$, $f'(x) < 0$ when $-2 < x < 2$, $f''(x) < 0$ when $x < 0$, $f''(x) > 0$ when $x > 0$

Sketch each of the following curves and show the local extrema values and the inflection points if found on the curves:

41 $f(x) = 4 - x^2$ 42 $f(x) = x^2 - 4x + 3$

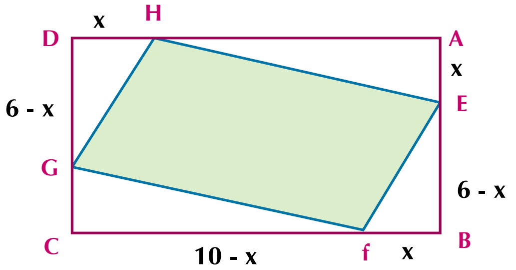
43 $f(x) = x(x + 3)^2$ 44 $f(x) = \frac{1}{3}x^3 - 4x + \frac{1}{3}$

45 $f(x) = 2x^3 - 9x^2 + 12x - 2$ 46 $f(x) = 5 + 3x - x^3$

47 Identify the values of a, b, c and d to the curve $y = ax^3 + bx^2 + cx + d$ where it has a local maximum value at (0 , 6) and a local minimum value at (1 , 5), then sketch the curve.

48 A 20 - meter wire is to be shaped in the form of a rectangle. What are the dimensions which make the area as maximum as possible?

49 **In the opposite figure** A B C D is a rectangle :



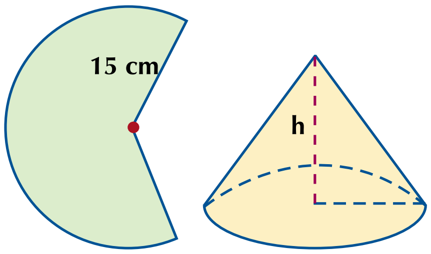
a Prove that the figure EFGH is a parallelogram whose area $A = 2x^2 - 16x + 60$

b Find the possible minimum value of the area A.

50 A circular sector -like piece of paper whose radius length of its circle is 15cm. The piece of paper is folded to form a right circular cone of height h cm. Show that the volume of the cone v cm^3 is given by the relation

$V = \frac{\pi}{3} h (225 - h^2)$,

then find the possible maximum volume of this cone.

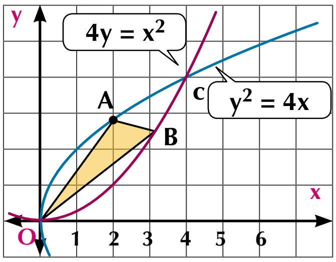


51 Find the height of a right circular cylinder that can be put in a hollow sphere of interior radius length 10cm when the lateral area of the cylinder is as maximum as possible.

52 Prove that for all the values of $x \in R$ the inequality is satisfied

$$\frac{x}{a x^2 + b} \leqslant \frac{1}{2\sqrt{ab}} \qquad \text{where } a > 0 ,\ b > 0$$

53 Let C be the intersection point of the two curves $y^2 = 4x$, $4y = x^2$, and point A lies on the curve $y^2 = 4x$ and its X coordinate equals 2 , point B(x, y) lies on the curve $4y = x^2$ between the two points O and C , find the possible maximum area of triangle O A B



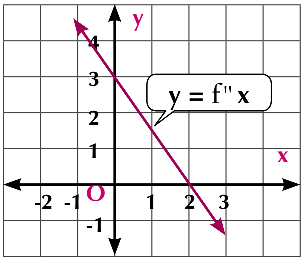
54 A covered tank for water is made up of a cuboid whose base is in the form of a square of side length 2x and height x and a right circular cylinder of diameter length 2x and height y is placed on the cuboid . If the total volume of the tank is 27 cm^3, find the value of x which makes the surface area of the tank as minimum as possible.

.

For more Activties and Excercises, Vist: www.sec3mathematics.com.eg

Accumulative test

1. The opposite figure shows the curve of f'' (x) of the function f, complete:



   a. The curve of f is convex upwards when $x \in$ ....
   b. The curve of f has inflection point when x = ...
   c. If f' (-1) = f' (5) = 0, then the function f has a local maximum value when x = ...
   d. f is decreasing for each $x \in$ ....

2. If $f'(x) = a x + b$ where $a < 0$
   a. Investigate the existance of a extrema value for the function f and show its type if found.
   b. Determine the increasing and decreasing intervals of the function f when a = -2 and b = 5

3. Determine the critical points and the increasing and decreasing intervals of the function f where $f(x) = \sqrt[3]{(x^2 - 1)^2}$

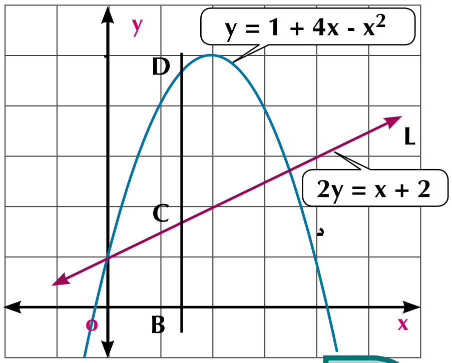
4. If $f(x) = x^3 - 6x^2 + 12x$, investigate the presence of critical points for the function f and determine the intervals of being convex upwards and downwards and the inflection points if found, then sketch the curve of the function f

5. Sketch the curve of the function f where y = f(x) if:
f is continuous and its domain is [-1 , 5] , f(1) = f(3) = 0 , f' (2) is not existed,
$f'(x) > 0$ when $x < 2$, $f'(x) < 0$ when $x > 2$, $f''(x) > 0$ when $x \neq 2$

6. If f is a differentiable function on $\mathbb{R}$ $f(x) = \begin{cases} 2x^2 + a x + b & \text{when } x \geqslant 1 \\ 3x - x^2 & \text{when } x < 1 \end{cases}$
   a. Find the values of the two constants a and b
   b. Determine the interval of being convex upwards and downwards and the inflection points if found.

7. From the set of all the ordered pairs (x, y) of the non- negative integer numbers which the sum of their projections is 5, find the ordered pair which makes the product of the square of the first projection and the cube of the second projection as maximum as possible .

8. **The opposite figure** shows the curve of the function f and the straight line L
If the straight line $\overleftrightarrow{BC}$ is parallel to y-axis and intersect the positive part of x-axis and the straight line L and the curve of f at points B, C and D respectively, find the coordinate of point B which makes CD as maximum as possible.



Unit Four

The Definite Integral and its Applications

Introduction

Have you ever noticed the basket maker doing a basket? The process of getting the parallel section side by side leads to the integration of his basket. This helped the scientists try to discover general methods to estimate the area of any plane region by dividing any plane area into very tiny areas, then adding the area of such tiny areas to estimate the required area. This had led to discover the science of integration and the symbol of the integration process $\int$ it is the first letter of the word "Sum" which means summation in this unit, you are going to know different methods to calculate the indefinite integration such as integration by substitition and integration by parts to find the set of antiderivatives for a continuous function on a given interval , then identify the definite integration through the fundamental theorem of calculus which links between the definite and indefinite integration and use the definite integration to find the area of a plane region or the volume of revolution solid. You will also identify some economic applications for the definite integration and use the mathematical modling to solve the mathematical and life problems .

Unit objectives

By the end of the unit and carrying out the involved activities, the students should be able to:

- Identify some methods of integration such as integration by substitution and by parts $\int x\, e^{x}\, dx$
- Identify the integration of the trigonometric functions and the fundamental integral table.
- Identify the definite integration (fundamental theorem of calculus) and deduce some of its properties.
    - $\int_{b}^{a} f(x)\, dx = -\int_{a}^{b} f(x)\, dx$
    - $\int_{a}^{a} f(x)\, dx = 0$
    - $\int_{a}^{b} k\, f(x)\, dx = k \int_{a}^{b} f(x)\, dx$
    - $\int_{a}^{b} [f(x) \pm g(x)]\, dx = \int_{a}^{b} f(x)\, dx \pm \int_{a}^{b} g(x)\, dx$
    - $\int_{a}^{b} f(x)\, dx + \int_{b}^{c} f(x)\, dx = \int_{a}^{c} f(x)\, dx$
    - $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$ where f is an even function.
    - $\int_{-a}^{a} f(x)\, dx = 0$ where f is an odd function.
- Use the definite integration to solve problem involving finding the area.
- Find the area of the plane region under the curve and on the x-axis where f (x) is non-negative for all the values of x in the domain using the definite integration .
- Find the area of the plane region included between two curves.
- Use the definite integration to solve problems involving finding the volume of a revolution surface around an axis of the coordinate axes .

Key terms

- Antiderivative
- Indefinite Integral
- Differential
- Integration by Substitution
- Integration by Parts
- Rule
- Trigonometric Function
- Definite Integral
- Fundamental theorem of calculus
- Areas in the plane
- Volumes of Revolution solids

Lessons of the unit

Lesson (4 - 1): Methods of integration.

Lesson (4 - 2): Integration of trigonometric functions.

Lesson (4 - 3): Definite integration.

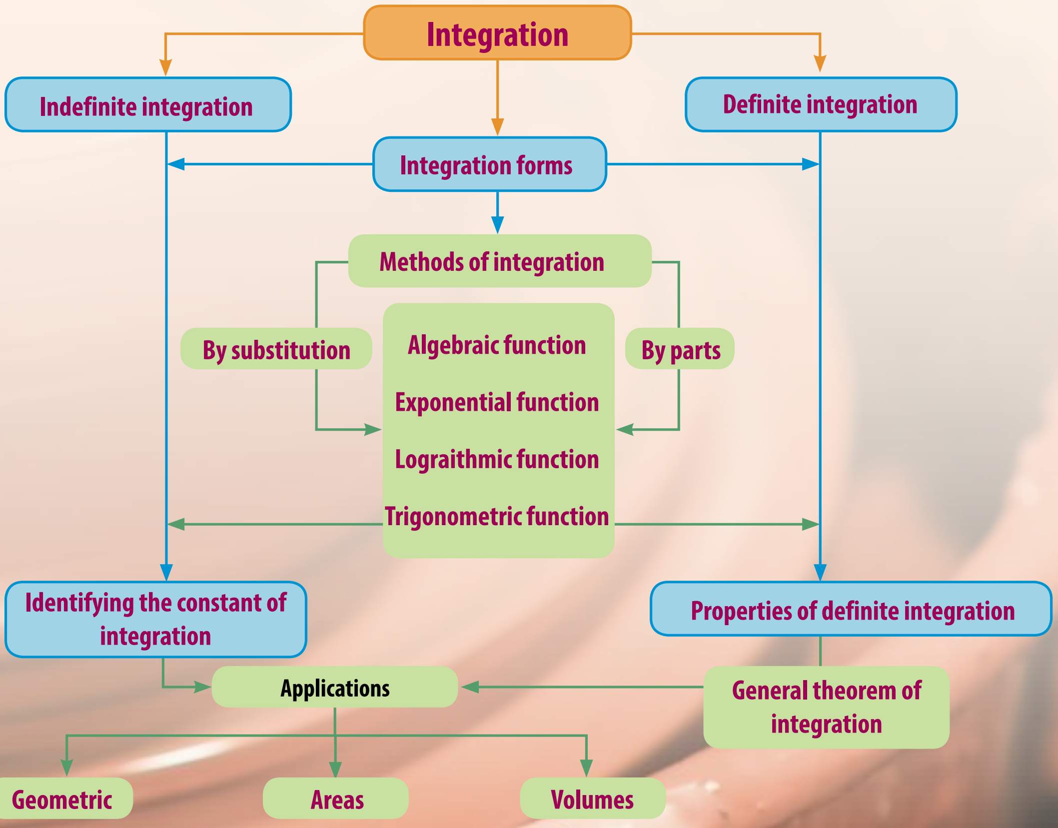
Lesson (4 - 4): Areas in the plane.

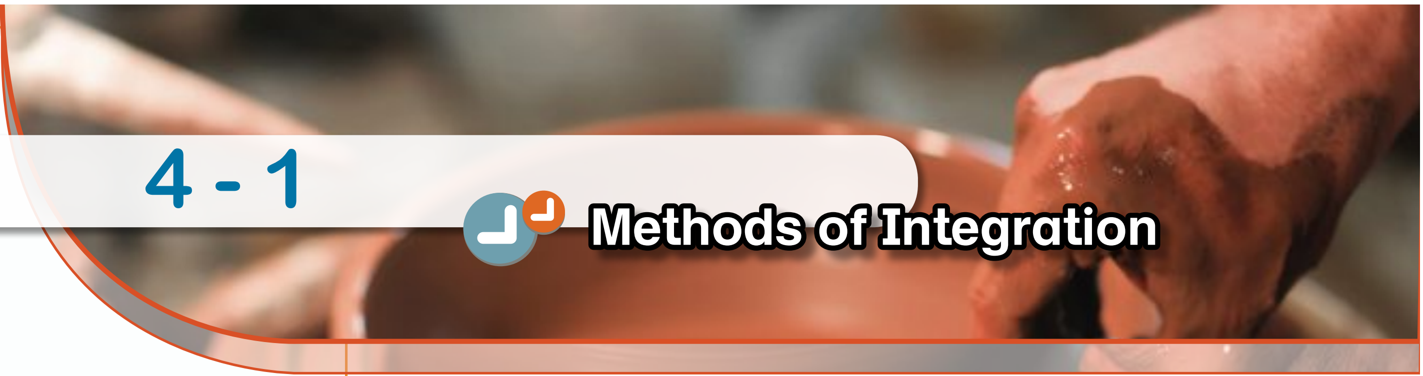
Lesson (4 - 5): Volumes of solids of revolution.

Materials

- Seientific calculator - Computer graphics.
- Internet).

Chart of the unit





4 - 1 Methods of Integration

You will learn

- Finding the original function of a given function
- Finding the differential of a function
- Calculating the integration by substitution
- Calculating the integration by parts

Key terms

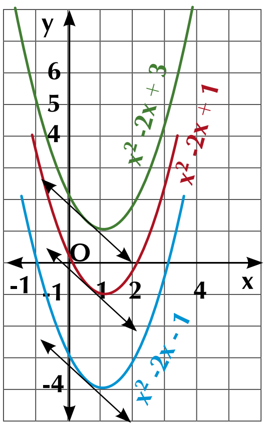
- Antiderivative
- Indefinite Integral
- Differential

Materials

- Scientific calculator.
- Computer graphics.

introduction

You have previously identified the anti derivative or the indefinite integration. It is an inverse operation to the process of differentiation . It is said that the function F is an antiderivative to the function f at the interval I if: $\frac{d}{dx}$ F(x) = f (x) for each x ∈ I



when we add any constant to the antiderivative F, (**it is known as the arbitrary constant**) then the antiderivative is represented by the set of the curves **y = F(x)** + **c** which are different from each other in the constant C and the slope of the tangent for any of them is equal .As a result these curves are parallel as shown in the opposite figure. The set of antiderivatives has been called the indefinite integration and denoted by the symbol: $\int f(x)\,dx$, then:

$$\int f(x)\,dx = F(x) + c$$

The indefinite integration has the next properties:

If f and g are two antiderivatives at the interval I , then:

1- $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

2- $\int k\,f(x)\,dx = k \int f(x)\,dx$ where K is a constant real number

Notice that:

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ where $n \neq -1$ and **c** is an arbitrary constant

So

$\int (3x^2 + 4x + 5)\,dx = x^3 + 2x^2 + 5x + c$

➲ The process of finding the antiderivatives requires to know the forms of standard integrations for some functions. But the integrations are needed to be found appear far from the standard integrations. This means that you are to identify other methods for integration such as the integration by substitution and integration by parts depending on the differential of the function.

Differentiations

If the function f is differentiable where y = f (x)

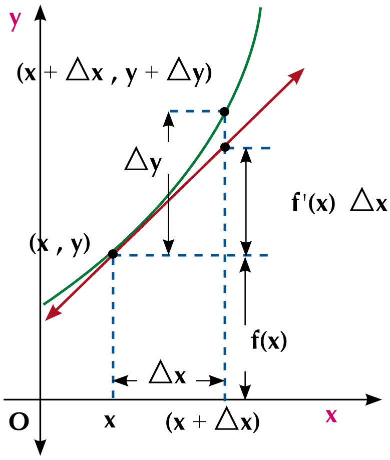
From the definition of the derivative:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{x}$$

$\therefore$ When $\Delta x \longrightarrow 0$, **then**: $\frac{\Delta y}{\Delta x} \longrightarrow f'(x)$

i.e.: $\frac{\Delta y}{\Delta x} \simeq f'(x)$ **when** $\Delta x \simeq 0$, $\Delta x \neq 0$

$\therefore \Delta y \simeq f'(x)\Delta x$ (by multiplying by $\times \Delta x$)



Definition

Let function f be differentiable on an open interval containing x , Δx is denoted for the change in x where $\Delta x \neq 0$, then

1- The differential of y (is denoted by the symbol d y) = $f'(x)\,\Delta x$

2- The differential of x (is denoted by the symbol d x) = Δx

So:

$dy = f'(x)\,dx$ It is a function in two variables x dx

If $y = x^3$ **Then:** $dy = 3x^2\,dx$

Example Differential of the funciton

1 Find the differential for each of the following:

a $y = \frac{x}{x-1}$ b $V = \frac{4}{3}\pi r^3$

c $y = z . \ell$ where each of z , ℓ is a function in x

Solution

a $\because dy = y'\,dx$, $y = \frac{x}{x-1} = 1 + \frac{1}{x-1} = 1 + (x-1)^{-1}$

$\therefore dy = \frac{-1}{(x-1)^2}\,dx$

b $\because dV = V'\,dr$ $\therefore dV = \frac{4}{3}\pi \times 3r^2\,dr = 4\pi r^2\,dr$

c $\because dy = (z \bullet \ell)'\,dx$

$= (z \bullet \ell' + \ell \bullet z')\,dx$

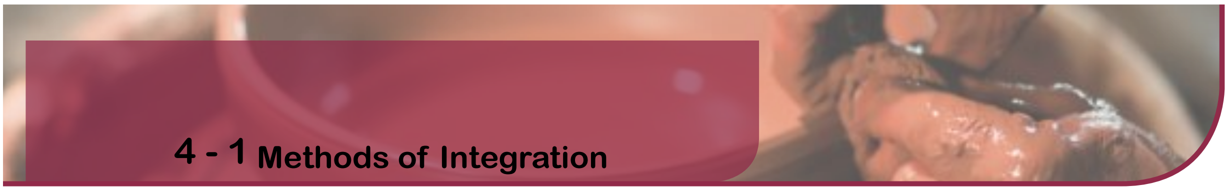
$= z \bullet \ell'\,dx + \ell \bullet z'\,dx$

$dy = z\,d\ell + \ell\,dz$

Notice

$d\ell = \ell'\,dx$

$dz = z'\,dx$



Try to solve

1. **Find the differential for each of :**

a) $y = (2x + 5)^4$ b) $y = e^{2x - 3}$

c) $y = \frac{z}{\ell}$ where z , ℓ are functions in the variable x

Critical thinking: If $x^2 + x^2 = 25$ **find:** d y in terms x , y, d x

The fundamental integrations (standard))

There is not a general method to find the integration of the different functions similar to the methods of finding the derivatives of these functions. Finding the integration of any function f is specified in searching for a function which its derivatives are the function f. This is closely related to your understanding to the fundamental derivatives of the functions which you have previously learned and we can summarize them in the following table:

Table of the fundamental derivatives of the functions and the corresponding standard integrations

$\frac{d}{dx}(x^n) = nx^{n-1}$ $n \in \mathbb{R}$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$ $n \in \mathbb{R} - \{1\}$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\, dx = \sin x + c$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\, dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$, $x \neq \frac{2n+1}{2}\pi$, $n \in Z$	$\int \sec^2 x\, dx = \tan x + c$, $x \neq \frac{2n+1}{2}\pi$, $n \in Z$
$\frac{d}{dx}(e^s) = e^x$	$\int e^x\, dx = e^x + c$
$\frac{d}{dx}(a^x) = a^x \ln a$ $a > 0 , a \neq 1$	$\int a^x\, dx = \frac{1}{\ln a} a^x + c$ $a > 0 , a \neq 1$
$\frac{d}{dx}(\ln ^{x}) = \frac{1}{x}$ $x > 0$	$\int \frac{1}{x}\, dx = \ln \lvert x \rvert + c$ $x \neq 0$

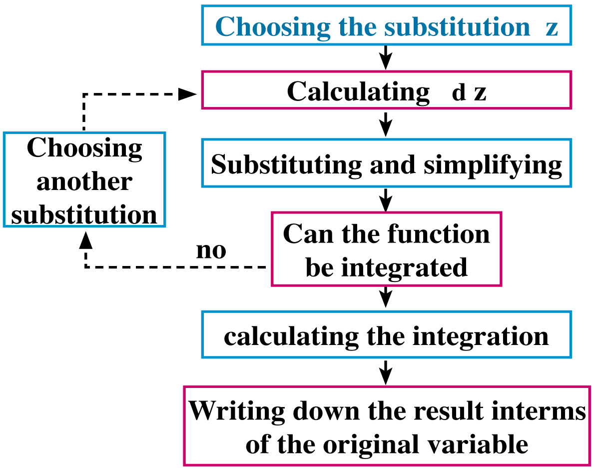
The Integration by Substitntion

It is one of the most essential methods of integration to find the integration of the product of two functions in the form: $\int f(g(x))\, g'(x)\, dx$

If $z = g(x)$ is a differentiable function then $dz = g'(x)\, dx$ and:

$\int f(g(x))\, g'(x)\, dx = \int f(z)\, dz$

➲ **To do the integration operation by substitution, we follow up the opposite diagram:**



Example Integration by substitution

(2) Find

a $\int x^3 (2x^4 - 7)^5 \, dx$ b $\int \frac{x+4}{(x^2+8x)^3} \, dx$

Solution

a By putting $z = 2x^4 - 7$ $\therefore \; dz = 8x^3 \, dx$

$$\int x^3 (2x^4 - 7)^5 \, dx = \frac{1}{8} \int (\underbrace{2x^4 - 7}_{z})^5 \, (\underbrace{8x^3 \, dx}_{dz})$$

$$= \frac{1}{8} \int z^5 \, dz = \frac{1}{8 \times 6} z^6 + c$$

$$= \frac{1}{48} (2x^4 - 7)^6 + c$$

b By putting $z = x^2 + 8x$

$\therefore dz = (2x + 8) \, dx = 2(x + 4) \, dx$

$$\int \frac{x+4}{(x^2+8x)^3} \, dx = \frac{1}{2} \int \frac{2(x+4) \, dx}{(x^2+8x)^3} = \frac{1}{2} \int \frac{1}{z^3} \, dz$$

$$= \frac{1}{2} \int z^{-3} \, dz = \frac{1}{2 \times -2} z^{-2} + c$$

$$= \frac{-1}{4(x^2+8x)^2} + c$$

Try to solve

(2) Find:

a $\int 3x (x^2 + 3)^4 \, dx$ b $\int \frac{x^2}{(x^3 - 4)^5} \, dx$

Example Integration by substitution

(3) Find

a $\int x (x + 4)^7 \, dx$ b $\int (x^2 + 5) \sqrt{x - 1} \, dx$

Solution

a By putting $z = x + 4$ $\therefore \; x = z - 4 , \, dx = dz$

$$\int x (x + 4)^7 \, dx = \int z^7 (z - 4) \, dz = \int (z^8 - 4z^7) \, dz \qquad \textbf{(substitution)}$$

$$= \frac{1}{9} z^9 - \frac{1}{2} z^8 + c \qquad \textbf{(integration)}$$

$$= \frac{1}{18} z^8 (2z - 9) + c \qquad \textbf{(simplification)}$$

$$= \frac{1}{18} (x + 4)^8 (2x - 1) \qquad \textbf{(substitution by z)}$$

b By putting $z^2 = x - 1$ **to simplify the form of integration** $\therefore\ x = z^2 + 1,\ dx = 2z\ dz$

$$\int (x^2+5)\sqrt{x-1}\ dx = \int [(z^2+1)^2+5] \times z \times 2z\ dz \qquad \textbf{(substitution)}$$

$$= \int [z^4 + 2z^2 + 6] \times 2z^2\ dz$$

$$= 2\int (z^6 + 2z^4 + 6z^2)\ dz \qquad \textbf{(simplification)}$$

$$= 2\left[\frac{1}{7} z^7 + \frac{2}{5} z^5 + 2z^3\right] + c \qquad \textbf{(integration)}$$

$$= \frac{2}{35} z^3 [5z^4 + 14 z^2 + 70] + c \qquad \textbf{(common factor)}$$

$$= \frac{2}{35} (x-1)^{\frac{3}{2}}\ [5(x-1)^2 + 14(x-1) + 70] + c \qquad \textbf{(substitution by z)}$$

$$= \frac{2}{35}\sqrt{(x-1)^3}\ (5x^2 + 4x + 61) + c$$

Try to solve

3 Find the following integrations:

a $\int x(2x-3)^4\ dx$ **b** $\int x^2 \sqrt[3]{3x+1}\ dx$

Example

Integration by sustitution

4 Find :

a $\int \sqrt{\frac{1+\sqrt{x}}{x}}\ dx$ **b** $\int 6x\, e^{x^2}\ dx$

Solution

a By putting $z = 1 + \sqrt{x}$ $\therefore\ \sqrt{x} = z - 1,\ x = (z-1)^2$

$$dx = 2(z-1)\ dz$$

$$\int \sqrt{\frac{1+\sqrt{x}}{x}}\ dx = \int \sqrt{\frac{z}{(z-1)^2}} \times 2(z-1)\ dz \qquad \textbf{(substitution)}$$

$$= \int 2z^{\frac{1}{2}}\ dz = 2 \times \frac{2}{3} z^{\frac{3}{2}} + c \qquad \textbf{(integration)}$$

$$= \frac{4}{3}\left(\sqrt{(1+\sqrt{x})^3}\right) + c \qquad \textbf{(substitution by z)}$$

b By putting $z = x^2$ $\therefore\ dz = 2x\ dx$

$$\int 6x\, e^{x^2}\ dx = 3\int e^{x^2}(2x\ dx)$$

$$= 3\int e^z\ dz = 3e^z + c \qquad \textbf{(substitution and integration)}$$

$$= 3e^{x^2} + c \qquad \textbf{(substitution by z)}$$

Try to solve

4 Find :

a $\int \frac{x}{\sqrt{1-3x^2}}\ dx$ **b** $\int (3-x)\, e^{6x-x^2}\ dx$

Example Integration by substitution

5 Find

a $\int \frac{5x}{3x^2 - 1}\, dx$ b $\int \frac{\sqrt{\ln x}}{x}\, dx$

Solution

a By putting $z = 3x^2 - 1$ $dz = 6x\, dx$

$$\int \frac{5x}{3x^2-1}\, dx = \frac{5}{6}\int \frac{6x\,dx}{3x^2-1} = \frac{5}{6}\int \frac{1}{z}\, dz \quad \textbf{(substitution)}$$

$$= \frac{5}{6}\ln|z| + c = \frac{5}{6}\ln|3x^2-1| + c \quad \textbf{(substitution and integration)}$$

b By putting $z = \ln x$ $\therefore\ dz = \frac{1}{x}\, dx$

$$\int \frac{\sqrt{\ln x}}{x}\, dx = \int \sqrt{\ln x}\left(\frac{1}{x}\, dx\right) = \int z^{\frac{1}{2}}\, dz \quad \textbf{(substitution)}$$

$$= \frac{2}{3} z^{\frac{3}{2}} + c$$

$$= \frac{2}{3}[\ln (x)]^{\frac{3}{2}} + c \quad \textbf{(substitution and integration)}$$

Try to solve

5 Find :

a $\int \frac{e^{2x}}{e^{2x} + 3}\, dx$ b $\int \frac{1}{x(\ln x)^2}\, dx$

Critical thinking: Use the integration by substitution to prove the correctness of the following integration rules :

1- $\int [f(x)]^n\, f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ where $n \neq -1$

2- $\int \frac{f'(x)}{f(x)}\, d(x) = \ln|f(x)| + c$ where $f(x) \neq 0$

Integration by Parts

If y and z are two functions in the variable x and differentiable , then:

$$\frac{d}{dx}(yz) = y\frac{dz}{dx} + z\frac{dy}{dx}$$

By integrating both sides with respect to x

$$\int \frac{d}{dx}(yz)\, dx = \int y \frac{dz}{dx}\, dx + \int z \frac{dy}{dx}\, dx$$

$$yz = \int y\, dz + \int z\, dy$$

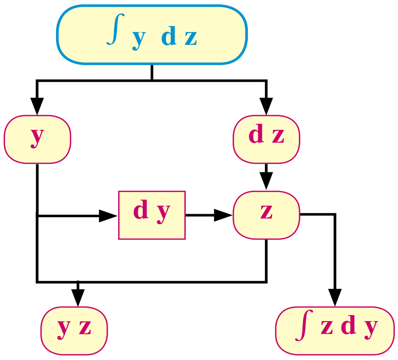
i.e : $\int y\, dz = yz - \int z\, dy$

Remember

$dz = \frac{dz}{dx}\, dx$

$dy = \frac{dy}{dx}\, dx$

The previous equation is called the rule of integration by parts. It is used to find the integration of the product of two functions each is not a derivative to another by a proper choice for each of y and z such that the integration can be calculated by the left side by a method easier than calculating the integration by the right side. The opposite diagram is to be followed up as shown in the following example :



Example Integration by parts

6 Find:

a $\int x\, e^{x}\, dx$ **b** $\int x^2 e^{x}\, dx$

Solution

a To find $\int x\, e^{x}\, dx$:

Let: $y = x$, $dz = e^{x}\, dx$

$\therefore\ dy = dx$ $z = \int e^{x}\, dx = e^{x}$

$\because\ \int y\, dz = yz - \int z\, dy$

$\therefore\ \int x\, e^{x}\, dx = x e^{x} - \int e^{x}\, dx = x e^{x} - e^{x} + c = e^{x}(x-1) + c$

Important note: Adding a constant to function z doesnot change the result (prove that)

b To find $\int x^2 e^{x}\, dx$:

Let: $y = x^2$, $dz = e^{x}\, dx$

$dy = 2x\, dy$ $z = \int e^{x}\, dx = e^{x}$

$\int x^2 e^{x}\, dx = x^2 e^{x} - \int 2x\, e^{x}\, dx$

$= x^2 e^{x} - 2\int x\, e^{x}\, dx$

$\int x\, e^{x}\, dx = e^{x}(x-1) + c$ from a

$= x^2 e^{x} - 2\, e^{x}(x-1) + c$

$= e^{x}[x^2 - 2x + 2] + c$

Try to solve

6 Find :

a $\int x\, e^{-2x}\, dx$ **b** $\int x^2 e^{x+3}\, dx$

Notice that:

Choosing y and dz is based on:

1- d y is simpler than y **2-** d z is easier than integration

Example Integration by parts

7 Find

a $\int \ln x \, dx$ b $\int x \ln x \, dx$

Solution

a **Let :** $y = \ln x$ $dz = dx$

$dy = \frac{1}{x} dx$ $z = \int dx = x$

$\int \ln x \, dx = x\ln x - \int x \times \frac{1}{x} dx$

$= x\ln x - x + c = x(\ln x - 1) + c$

b **Let:** $y = \ln x$ $dz = x\, dx$

$dy = \frac{1}{x} dx$ $z = \int x\, dx = \frac{1}{2} x^2$

$\int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \times \frac{1}{x} dx$

$= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + c = \frac{1}{2} x^2 (\ln x - 1) + c$

Try to solve

7 Find:

a $\int \ln (x + 1) \, dx$ b $\int (\ln x \div \sqrt{x}) \, dx$

Example Integration by parts

8 Find:

a $\int \frac{x e^x}{(x+1)^2} dx$ b $\int \frac{4x}{\sqrt[3]{2x+1}} dx$

Solution

a Notice that $(x + 1)^{-2}$ is easier in the integration

By putting $y = x e^x$ $dz = (x + 1)^{-2} dx$

$dy = (x e^x + e^x) dx$ $z = \frac{-1}{x+1}$

$\int \frac{x e^x}{(x+1)^2} dx = -\frac{x e^x}{x+1} - \int \frac{-1}{x+1} \times e^x (x + 1) \, dx$

$= -\frac{x e^x}{x+1} + \int e^x dx$

$= \frac{-x e^x + e^x(x + 1)}{x + 1} + c = \frac{e^x}{x + 1} + c$

b By putting $y = 4x$ $\quad dz = (2x+1)^{-\frac{1}{3}}\, dx$

$dy = 4\, dx$ $\quad z = \frac{3}{2 \times 2}\,(2x+1)^{\frac{2}{3}}$

$$\int \frac{4x}{\sqrt[3]{2x+1}}\, dx = 3x\,(2x+1)^{\frac{2}{3}} - \int \frac{3}{4}\,(2x+1)^{\frac{2}{3}} \times 4\, dx$$

$$= 3x\,(2x+1)^{\frac{2}{3}} - \frac{3 \times 3}{5 \times 2}\,(2x+1)^{\frac{5}{3}} + c$$

$$= \frac{3}{10}\,(2x+1)^{\frac{2}{3}}\,[10x - 3\,(2x+1)] + c$$

$$= \frac{3}{10}(2x+1)^{\frac{2}{3}}\,(4x-7) + c$$

Try to solve

8 Find :

a $\int \frac{3x+5}{e^{2x}}\, dx$ **b** $\int \frac{x}{\sqrt{2x+3}}\, dx$

Critical thinking: can you find $\int \frac{4x}{\sqrt[3]{2x+1}}\, dx$ using the integration by substitution? Explain.

Some applications of the indefinite integration

If we know that the function g gives the slope of the tangent (or the function of the profit margin or the rate of change of a function) at any point on the curve of the function f, then we can know the function F from the operation of the indefinite integration of the function g where: $f(x) = \int g(x)\, dx$

it is noticed that the integration does not give only one function since it contains an arbitrary constant which can be determined from the given data.

Example The equation of the curve of the function

9 If the slope of the tangent to the curve of the function f at any point (x , y) lying on it is given by the relation $g(x) = \frac{x\,e^{x}}{(x+1)^2}$, find the equation of the curve if it passes through point (1 , 2 e) .

Solution

Let the equation of the curve of the function be y = f(x) $\quad \therefore f(x) = \int g(x)\, dx$

$\therefore f(x) = \int \frac{x e^{x}}{(x+1)^2}\, dx = \frac{e^{x}}{x+1} + c$ [from the solution of example 9 (a)]

$\because$ The curve of f passes through point (1 , 2 e) it satisfies the equation

$\therefore 2e = \frac{e}{(1+1)^2} + c$

$\therefore c = \frac{7e}{4}$ and $f(x) = \frac{e^{x}}{x+1} + \frac{7}{4}e$

Try to solve

9) Find the equation of the curve passing through point (0 , 1) and its slope of tangent at point (x , y) lying on it equals $x\sqrt{x^2+1}$

Exercises 4-1

Choose the correct answer

1) $\int x(x^2+3)^5\,dx$ equals

a) $\frac{1}{6}(x^2+3)^6+c$ b) $\frac{1}{12}(x^2+3)^6+c$ c) $\frac{1}{4}(x^2+3)^4+c$ d) $\frac{1}{8}(x^2+3)^4+c$

2) If $\int(2x+3)\ln x\,dx = yz - \int z\,dy$, then y z equals

a) $2x\ln x$ b) $(2x+3)\ln x$ c) $\frac{1}{2}(2x+3)\ln x$ d) $x(x+3)\ln x$

3) If $\int(2x-1)e^{2x+3}\,dx = yz - \int z\,dy$, then $\int z\,dy$

a) $e^{2x+3}+c$ b) $\frac{1}{2}e^{2x+3}+c$ c) $-e^{2x+3}+c$ d) $-\frac{1}{2}e^{2x+3}+c$

Use the proper substitution to find the following integrations:

4) $\int x(x-2)^4\,dx$
5) $\int x^2(x-2)^3\,dx$
6) $\int x^3(x^2-1)^5\,dx$
7) $\int x\sqrt{x+4}\,dx$
8) $\int(x^2-1)\sqrt{x+1}\,dx$
9) $\int x^5(x^2+3)^4\,dx$
10) $\int \frac{x}{3x^2+2}\,dx$
11) $\int \frac{x}{x+1}\,dx$
12) $\int \frac{x+1}{\sqrt{x-1}}\,dx$
13) $\int \frac{x^2}{\sqrt{2x-1}}\,dx$
14) $\int x e^{-x^2}\,dx$
15) $\int \frac{e^{-x}-1}{e^{-x}+x}\,dx$
16) $\int \ln 5x\,\frac{dx}{x}$
17) $\int (\ln x)^3\,\frac{dx}{x}$
18) $\int \frac{1}{x\ln x}\,dx$

Use the proper parts to find the following integrations :

19) $\int 4x e^{2x}\,dx$
20) $\int x^3 e^{x^2}\,dx$
21) $\int \frac{x}{e^{2x}}\,dx$
22) $\int x^3\ln x\,dx$
23) $\int \ln x^2\,dx$
24) $\int (\ln x)^2\,dx$
25) $\int \ln x\,\frac{dx}{x^3}$
26) $\int (x+1)^2 e^{2x}\,dx$
27) $\int x(\ln x)^2\,dx$

Answer the following:

28) Find the equation of the curve passing through point A (2 , 3), and the slope of the normal on it at any point (x , y) is 3- x.

29) If the slope of a tangent to a curve at point (x , y) lying on it is $x\sqrt{x+1}$, find the equation of the curve known that the curve passes through point $(0, \frac{11}{15})$

30) Find the equation of the curve y = f (x) if $\frac{d^2y}{dx^2} = ax+b$ where a and b are two constants and the curve has an inflection point at point (0 , 2) and a local minimum value at point (1 , 0) then find the local maximum value to this curve.

4 - 2 Integral of Trigonometric Functions

You will learn

- Rules of the integration of the trigonometric fucntions

Key terms

- Rule
- Trigonometric function
- Table of integrals

Materials

- Scientific calculator

Think and discuss

Finding the integration of any function is specified to search for another function F which if you derive its first derivative , you get the function f , i.e. : $f(x) = \frac{d}{dx} F(x)$ then:

$\int f(x)\, dx = F(x) + c$ where c is an arbitrary constant.

Show which of the following relations is true:

a $\int \cos x\, dx = \sin x + c$ **b** $\int \sin x\, dx = \cos x + c$

c $\int \sec^2 x\, dx = \tan x + c$ **d** $\int \csc^2 x\, dx = \cot x + c$

Note that: Your understanding to the first derivative of the trigonometric functions will help you find the integrations of such functions .

From your previous learning to the derivatives of the trigonometric functions (as in Remember). The next table shows the rules of the indefinite integration of some trigonometric functions . Compare the derivatives of the trigonometric functions and deduce their integrations, then complete the table.

Idndefinite integration
$\int \sin x\, dx = -\cos x + c$
$\int \cos x\, dx = \sin x + c$
$\int \sec^2 x\, dx = \tan x + c$
$\int \csc^2 x\, dx =$
$\int \sec x \tan x\, dx =$
$\int \csc x \cot x\, dx =$

Remember

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

Check your deduction using the definition of the antiderivative .

Example Table of integration

1 Find the following integrations:

a $\int (\sin x + \cos x)\, d x$

b $\int \sec x\, (\sec x - \tan x)\, d x$

c $\int \frac{1}{1 - \cos^2 x}\, d x$

d $\int \frac{\cos x}{1 - \cos^2 x}\, d x$

Solution

a $\int (\sin x + \cos x)\, d x = \int \sin x\, d x + \int \cos x\, d x$
$= - \cos x + \sin x + c$

b $\int (\sec x - \tan x) \sec x\, d x = \int \sec^2 x\, d x - \int \sec x \tan x\, d x$
$= \tan x - \sec x + c$

c $\int \frac{1}{1 - \cos^2 x}\, d x = \int \frac{1}{\sin^2 x}\, d x = \int \csc^2 x\, d x = - \cot x + c$

d $\int \frac{\cos x}{1 - \cos^2 x}\, d x = \int \frac{\cos x}{\sin^2 x}\, d x = \int \frac{1}{\sin x} \times \frac{\cos x}{\sin x}\, dx$
$= \int \csc x \cot x\, d x = - \csc x + c$

Remember

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\cot^2 x + 1 = \csc^2 x$

Try to solve

1 Find:

a $\int (\sin x + \sec^2 x)\, d x$

b $\int \sec x\, (\cos^2 x + \tan x)\, dx$

c $\int \csc x\, (\cot x - \csc x)\, d x$

d $\int \frac{\cos^2 x}{1 - \sin x}\, d x$

Important note:

You know that : $\int x^n\, d x = \frac{x^{n+1}}{n+1} + c$ **(1)** , $\int \sin x\, d x = - \cos x + c$ **(2)**

To generalize the previous results or all the standard forms of the integration , we notice that when we add a constant to the independent variable x, it doesnot affect the formula of the integration. Furthermore, when we multiply x by the coefficient a , the integration keeps its previous formula but it is devided by this coefficient (factor).

So, we find that the previous form for each of 1 and 2 are:

$\int (a x + b)^n\, d x = \frac{(a x + b)^n}{a(n + 1)} + c$

$, \int \sin (a x + b)\, d x = - \frac{1}{a} \cos (a x + b) + c$

Example The standard forms of the integration

(2) Find :

a $\int \sin(2x - 5)\, dx$

b $\int \sec^2(5 - 3x)\, dx$

c $\int \csc^2\left(\frac{x+3}{2}\right) dx$

d $\int \sec 2x \tan 2x\, dx$

Solution

a $\int \sin(2x - 5)\, dx = -\frac{1}{2}\cos(2x - 5) + c$

b $\int \sec^2(5 - 3x)\, dx = -\frac{1}{3}\tan(5 - 3x) + c$

c $\int \csc^2\left(\frac{1}{2}x + \frac{3}{2}\right) dx = -2\cot\left(\frac{x+3}{2}\right) + c$

d $\int \sec 2x \tan 2x\, dx = \frac{1}{2}\sec 2x + c$

Try to solve

(2) Find :

a $\int (\sin 3x - \sec^2 2x)\, dx$

b $\int 6\sec 2x\,(\tan 2x + \cos^2 2x)\, dx$

c $\int [1 + \cot^2(3x - 1)]\, dx$

d $\int \frac{\sin^2(2x - 3)}{1 - \cos(2x - 3)}\, dx$

Example Applications

(3) If the slope of the tangent to a curve at any point (x , y) on it is given by the relation:

$\frac{dy}{dx} = \sin x \cos x$, find the equation of the curve given that it passes through point $\left(\frac{\pi}{6}, 1\right)$

Solution

$\because$ The slope of the tangent at any point: $\frac{dy}{dx} = \sin x \cos x$

$\therefore$ The equation of the curve: $y = \int \frac{dy}{dx} \cdot dx = \int \sin x \cos x\, dx$

$$= \frac{1}{2}\int 2\sin x \cos x\, dx = \frac{1}{2}\int \sin 2x\, dx$$

$$= \frac{1}{2} \times -\frac{1}{2}\cos 2x + c = -\frac{1}{4}\cos 2x + c$$

$\because$ The curve passes through point $\left(\frac{\pi}{6}, 1\right)$

$\therefore$ It satisfies the equation

$1 = -\frac{1}{4}\cos\left(2 \times \frac{\pi}{6}\right) + c$ $\therefore c = 1 + \frac{1}{8} = \frac{9}{8}$

$\therefore$ The equation of the function is : $y = -\frac{1}{4}\cos 2x + \frac{9}{8}$

Remember

$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

Try to solve

(3) Find the equation of the curve passing through point (1 , -2) and its slope of tangent at any point (x , y) on it is:

$\frac{dy}{dx} = 3\pi \cos \pi x - 2\pi \sin \pi x$

Exercises 4-2

Choose the correct answer

1. If $\frac{dy}{dx} = \csc^2 x$, y = 2 when $x = \frac{\pi}{4}$, then y equals
   - a) $-2 - \cot x$
   - b) $2 - \cot x$
   - c) $-3 - \cot x$
   - d) $3 - \cot x$

2. $\int 2\cos^2 x\, dx$ equals
   - a) $x + \frac{1}{2}\sin 2x + c$
   - b) $x + 2\sin 2x + c$
   - c) $x - \frac{1}{2}\sin 2x + c$
   - d) $x - \sin 2x + c$

3. $\int \sec^4 x \tan x\, dx$ equals
   - a) $\frac{1}{5}\sec^5 x + c$
   - b) $\frac{1}{4}\sec^4 x + c$
   - c) $\frac{1}{3}\tan^3 x + c$
   - d) $-\frac{1}{3}\tan^3 x + c$

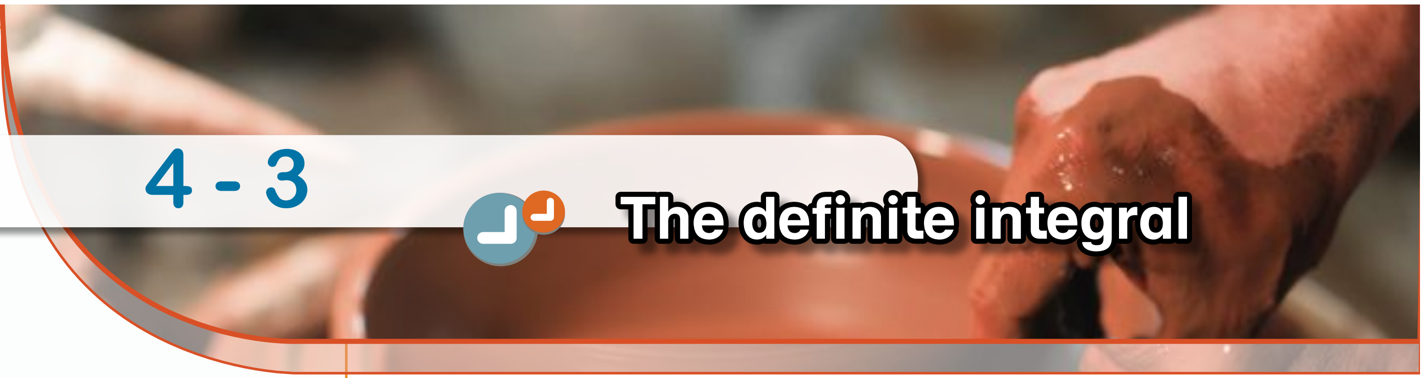
4. $\int (3x + 2)\sin x\, dx$ equals
   - a) $(3x + 2)\cos x + 3\sin x + c$
   - b) $-(3x + 2)\cos x + 3\sin x + c$
   - c) $(3x + 1)\cos x + 2\sin x + c$
   - d) $-(x + 1)\cos x - 2\sin x + c$

Find the following integrations:

5. $\int \sqrt{3}\cos x\, dx$
6. $\int \csc^2(2x + 3)\, dx$
7. $\int \tan x \cos x\, dx$
8. $\int \sin x \cos x\, dx$
9. $\int (1 + \tan^2 x)\cos^2 x\, dx$
10. $\int (\sin x - \cos x)^2\, dx$
11. $\int \cos^3 x \sin x\, dx$
12. $\int \sin^5 x \cos x\, dx$
13. $\int x^2 \cos(x^3 + 5)\, dx$
14. $\int (3 + \sin x)^5 \cos x\, dx$
15. $\int \sec^5 x \tan x\, dx$
16. $\int \frac{\cos^3 x - 4}{\cos^2 x}\, dx$
17. $\int (\cos^2 x + \cos x)\, dx$
18. $\int (\cos x + \sec x)^2\, dx$
19. $\int (\tan^2 x + 2\sin^2 x)\, dx$

Answer the following

20. If $\frac{dy}{dx} = 7 - \sin^2 x$, find y in terms x if y = 5 when x = 0
21. Find the equation of the curve passing through point $(\frac{\pi}{2} , \frac{\pi^2}{4} + 9)$ if its slope of tangent at any point (x , y) on it is given by the following relation: $m = 2x + \frac{1}{2}\sec^2 \frac{x}{2}$
22. The curve to the slope of the tangent at any point equals $-a\csc^2 x$ where a is a constant. If the curve passes through the two points $(\frac{\pi}{4} , 5)$, $(\frac{3\pi}{4} , 1)$, find the equation of the curve.



4 - 3 The definite integral

You will learn

- The concept of the definite integration.
- Using the fundamental theorem of calculus to find the definite integration
- Some properties of the definite integration

Key terms

- Definite Integral

Materials

- Scientific calculator
- Graphical programs

Think and discuss

If $y = f(x)$, and the slope of the tangent at any point (x, y) on the curve is :

$$\frac{dy}{dx} = f'(x) = 2x + 3$$

Can you determine a definite value for each of f(3), f(5) and f(5) - f(3)? Explain .

Notice that

1 From the definition of the indefinite integral:

$$y = \int f'(x)\, dx = f(x) + c$$

where c is an arbitrary constant which independent of x and it is essential to keep it in the integration to be comprehensive for all the functions whose rates of changes are f(x). As a result, the indefinite integral doesnot give a definite result corresponding a definite value of the variable x .

2 If the value of the integration when $x = a$ is $f(a) + c$ and its value when $x = b$ is $f(b) + c$

$\therefore$ The difference between the two values of the integration when $x = a$ and $x = b$

equals $f(b) - f(a)$ it is a definite value (whatever the value of the constant C is) it is denoted by the symbol $\int_a^b f'(x)\, dx$

where:

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

This form is called the definite integral.

Learn

The Foundamental Theorem of Calculus

If the function f is continuous on the interval [a , b] , and F is any anti derivative to the function f on the same interval , then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Notes:

1. $\int_a^b f(x)\,dx$ is called the definite integral and read as the definite integral of f(x) with respect to x from a to b, it is a real number whose value is based on:
   - **a** The upper and lower terms of the definite integration i.e. on the two numbers a and b respectively
   - **b** The rule of the function f

   The symbol of the variable x can be replaced by another symbol without affecting the amount of the integration, i.e:

   $$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz \ldots\ldots$$

2. F(b) - F(a) is expressed in the form $[F(x)]_a^b$ or $F(x)\Big|_a^b$

3. You can get the definite integral by finding the indefinite integral with disregarding the constant of the integration. Why? then , you can substitute the variable by the two terms of the integration.

4. All the rules of the indefinite integrations and the table of the standard integrations are to be applied to find the value of the definite integration of a continuous function , if f and g are two continuous functions on the interval [a, b]

Then:

- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$ where $k \in \mathbb{R}$

Example Calculating the value of the definite integral

1. Find the definite integration to the function f from x = -2 to x = 4 where $f(x) = 3x^2 - 2$

Solution

The function f is a continuous polynomial on $\mathbb{R}$

$$\therefore \int_{-2}^{4} (3x^2 - 2)\,dx = [x^3 - 2x]_{-2}^{4}$$
$$= [(4)^3 - 2(4)] - [(-2)^3 - (-4)]$$
$$= 64 - 8 + 8 - 4 = 60 \in \mathbb{R}$$

Try to solve

1 Find the value for each of the following:

a $\int_{-1}^{2} (2x + 3)\,dx$ b $\int_{0}^{5} \frac{3}{\sqrt{n+4}}\,dn$ c $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,d\theta$

Theorem If function f is continuous on the interval [a, b], then it is integrable on this interval.

Critical thinking

What is the difference between the definite and indefinite integration? Explain.

Properties of definite integral

If the function f is continuous on the interval [a, b], $c \in]a, b[$, then:

1. $\int_{b}^{a} f(x)\,dx = -\int_{a}^{b} f(x)\,dx$
2. $\int_{a}^{a} f(x)\,dx = 0$
3. $\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$

Example properties of the definite integral

2 If the function f is continuous on $\mathbb{R}$ and $\int_{1}^{3} f(x)\,dx = 6$, $\int_{5}^{3} f(x) = -14$, find $\int_{1}^{5} f(x)\,dx$

Solution

$\because$ f is continuous on $\mathbb{R}$ and x = 3 divides the interval [1, 5]

$$\therefore \int_{1}^{5} f(x)\,dx = \int_{1}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx \quad \text{property (3)}$$
$$= \int_{1}^{3} f(x)\,dx - \int_{5}^{3} f(x)\,dx \quad \text{property (1)}$$
$$= 6 - (-14) = 20$$

Try to solve

2 If the function f is continuous on $\mathbb{R}$ and $\int_{-1}^{4} f(x)\,dx = 255$, $\int_{2}^{-1} f(x)\,dx = -15$, find: $\int_{2}^{4} f(x)\,dx$

Example Calculating the value of the definite integration

3 Find $\int_0^5 |x - 3| \, dx$

Solution

From the definition of the modulus function , we find that $|x - 3| = \begin{cases} -(x - 3) & \text{when } x < 3 \\ x - 3 & \text{when } x \geq 3 \end{cases}$

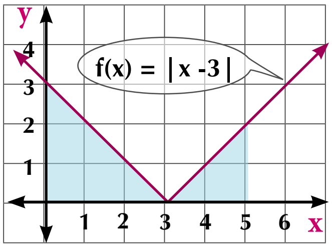
f is continuous when x = 3

$$\int_0^5 |x - 3| \, dx = \int_0^3 |x - 3| \, dx + \int_3^5 |x - 3| \, dx$$

$$= \int_0^3 (3 - x) \, dx + \int_3^5 (x - 3) \, dx$$

$$= [3x - \frac{x^2}{2}]_0^3 + [\frac{x^2}{2} - 3x]_3^5$$

$$= (9 - \frac{9}{2}) + (\frac{25}{2} - 15 - \frac{9}{2} + 9) = \frac{13}{2}$$



Notice that the colored area equals $\frac{13}{2}$ square units

Try to solve

3 Find:

a $\int_{-4}^2 |x + 1| \, dx$ b $\int_{-3}^3 |x^2 - 4| \, dx$

Example Calculating the value of the definite integration by substitution

4 Find the value of $\int_0^3 x\sqrt{x^2 + 3} \, dx$

Solution

You can get the definite integration by finding the indefinite integration first, then you can substitute the variable x using the two terms of the integration :

First:

to find $\int x\sqrt{x^2 + 3} \, dx$ put $z = x^2 + 3$ $\therefore dz = 2x \, dx$

$\therefore \int x\sqrt{x^2 + 3} \, dx = \frac{1}{2}\int \sqrt{x^2 + 3} \, (2x \, dx) = \frac{1}{2}\int \sqrt{z} \, dz$ (substitution)

$= \frac{1}{2}\int z^{\frac{1}{2}} \, dz = \frac{1}{2} \times \frac{2}{3} z^{\frac{3}{2}} + c$ (integration)

$= \frac{1}{3}\sqrt{(x^2 + 3)^3} + c$ (substitution by z)

Second:

$\int_0^3 \sqrt{x^2 + 3} \, dx = [\frac{1}{3}\sqrt{(x^2 + 3)^3}]_0^3$

$\frac{1}{3}[\sqrt{(12)^3} - \sqrt{(3)^3}] = 7\sqrt{3}$

Try to solve

4) Find:

a) $\int_{0}^{5} x\sqrt{25 - x^2}\, d\,x$ b) $\int_{-2}^{2} x^3\sqrt{x^2 + 3}\, d\,x$

Notice that

1. Example no 4 can be solved directly by finding the values of z corresponding to the values of the two terms of the integration (x = 0 , x = 3)

when x = 0 $\therefore$ z = 3 , when x = 3 z = 12

$$\therefore \int_{0}^{3} x\sqrt{x^2 + 3}\, d\,x = \frac{1}{2}\int_{3}^{12} z^{\frac{1}{2}} d\,x = [\frac{1}{3} z^{\frac{3}{2}}]_{3}^{12}$$

$$= \frac{1}{3}[\sqrt{(12)^3} - \sqrt{(3)^3}] = 7\sqrt{3}$$

2. In **try to solve** 4 b: $f(x) = x^3\sqrt{x^2 + 3}$ **an odd function**

In **try to solve** 3b: $f(x) = |x^2 - 4|$ **an even function**

The odd and even functions in the definite integration have the following properties:

> **1.** If function f is continuous and odd on the interval [-a , a], then:
>
> $$\int_{-a}^{a} f(x)\, d\,x = \text{zero}$$
>
> **2.** If function f is continuous and even on the interval [-a , a], then:
>
> $$\int_{-a}^{a} f(x)\, d\,x = 2\int_{0}^{a} f(x)\, d\,x$$

Use the properties above to check your answer in **try to solve** 3 and 4

Example The definite integration of odd and even function

5) Find:

a) $\int_{-2}^{2} \frac{x^3 - 3x}{x^2 + 1} dx$ b) $\int_{-3}^{3} (x^2 - 1)\, d\,x$

Solution

a) f is a continuous function on $\mathbb{R}$

$$\because f(-x) = \frac{(-x)^3 - 3(-x)}{(-x)^2 + 1} = -\frac{x^3 - 3x}{x^2 + 1} = -f(x)$$

$\therefore$ **f is an odd function and :** $\int_{-2}^{2} \frac{x^3 - 3x}{x^2 + 1} = \text{zero}$

b f is a continuous polynomial function on $\mathbb{R}$

$\because$ $f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$

$\therefore$ f is an even function and: $\int_{-3}^{3} (x^2 - 1)\, dx = 2 \int_{0}^{3} (x^2 - 1)\, dx$

$= 2 [\frac{1}{3} x^3 - x]^3_0 = 2 \times 6 = 12$

Try to solve

5 Find

a $\int_{-3}^{3} \frac{x}{1 + x^2}\, dx$ **b** $\int_{-\pi}^{\pi} (4 + \pi \cos 2x)\, dx$

Critical thinking

1 What is the value of $\int_{-3}^{5} f(x)\, dx$ if the function f is continuous and odd on the interval $[-3 , 5]$, $\int_{3}^{5} f(x)\, dx = 9$?

2 what is the value of $\int_{-4}^{-2} f(x)\, dx$ if the function f is continuous and even on the interval $[-4, 4]$, $\int_{-4}^{4} f(x)\, dx = 20$ and $\int_{0}^{2} f(x)\, dx = 6$?

Exercises 4 - 3

Choose the correct answer

1 If $\int_{-2}^{3} f(x)\, dx = 12$, $\int_{-2}^{5} f(x)\, dx = 16$, then $\int_{3}^{5} f(x)\, dx$ equals:

a -28 **b** -4 **c** 4 **d** 28

2 If $f(x) = |x|$, then $\int_{-2}^{2} f(x)\, dx$ equals:

a -1 **b** zero **c** 2 **d** 4

Find the value for each of :

3 $\int_{-1}^{3} x^3\, dx$ **4** $\int_{1}^{3} (3x^2 - 2)\, dx$ **5** $\int_{0}^{4} (2x + 1)^{\frac{3}{2}}\, dx$

6 $\int_{4}^{0} \frac{dt}{\sqrt{8 - t}}$ **7** $\int_{0}^{2} x(x^2 - 3)^2\, dx$ **8** $\int_{0}^{2} x^2 \sqrt[3]{x^3 + 1}\, dx$

9 $\int_{-1}^{3} |x - 1|\, dx$ **10** $\int_{-4}^{0} x (x + 4)^3\, dx$ **11** $\int_{10}^{3} x \sqrt{x + 1}\, dx$

12 $\int_{0}^{1} 2 \sin \pi z\, dz$ **13** $\int_{0}^{\frac{\pi}{4}} \tan z \sec^2 z\, dz$ **14** $\int_{0}^{3} (2x - 7e^x)\, dx$

Answer the following :

(15) If $\int_{1}^{5} f(x)\,dx = 10$ and $\int_{1}^{5} g(x)\,dx = -2$, calculate the value of

a $\int_{1}^{5} [f(x) + g(x)]\,dx$ b $\int_{1}^{5} [f(x) - g(x)]\,dx$ c $\int_{5}^{1} 3\,g(x)\,dx$

(16) If function f is continuous on the interval [-4 , 4] and $\int_{0}^{4} f(x)\,dx = 3$, find the value of

a $\int_{0}^{4} [f(x) + 2]\,dx$ b $\int_{-4}^{4} f(x)\,dx$, f odd c $\int_{-4}^{4} f(x)\,dx$, f even

(17) If $f(x) = \begin{cases} 2 & \text{when } x < 2 \\ x & \text{when } x \leqslant 2 \end{cases}$, find $\int_{0}^{6} f(x)\,dx$

4 - 4 Areas in the Plane

Think and discuss

1. Calculate the colored area in each figure of the following geometric figures.

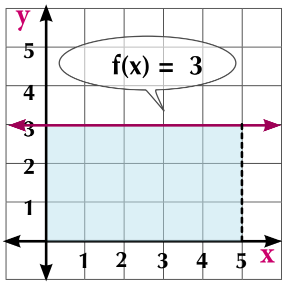


figure (1)

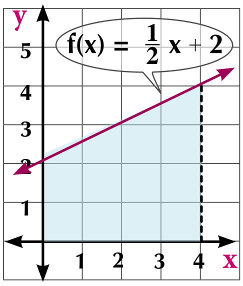


figure (2)

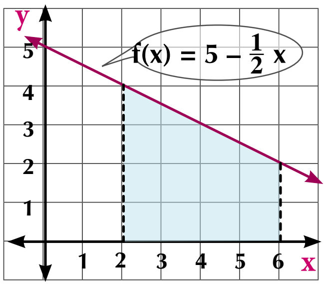


figure (3)

2. In each of the figures above, calculate $\int_a^b f(x)\,dx$ where f(x) is the equation of the curve and the two straight lines x = a , x = b bound the colored area.

3. Compare the area in each figure and the result of its definite integration. What do you infer?

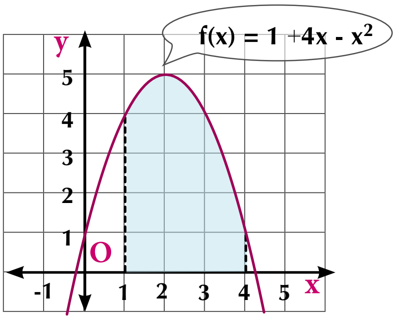
First: The area of a region bounded by the curve of the function f and x-axis in the interval [a , b]

Theorem If function f is continuous on the interval [a, b] and $f(x) \geqslant 0$ in this interval, A is the area of the region bounded by the curve of the function f, x-axis and the two straight lines x = a , x = b, then:

$$A = \int_a^b f(x)\,dx$$

Example — The area under the curve

(1) The opposite figure shows the curve of the function f where $f(x) = 1 + 4x - x^2$. Find the area of the region bounded by the curve of the function , x-axis and the two straight lines x = 1, x = 4



You will learn

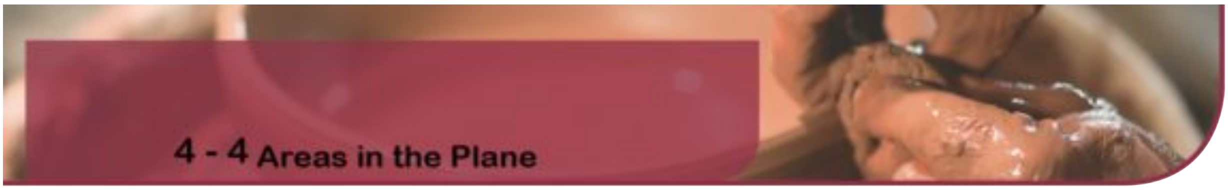
- Identify the area as a definite integration
- Find the area bounded by the curve of a function and x-axis on a closed interval
- Find the definite area between two intersecting curves

Key terms

- Area
- Square unit

Materials

- Scientific calculator
- Computer graphics



Solution

f is continuous on the interval [1, 4] and f(x) > 0 for each x ∈ [1, 4]

$\therefore A = \int_1^4 f(x)\,dx \qquad = \int_1^4 (1 + 4x - x^2)\,dx$

$= [x + 2x^2 - \frac{1}{3}x^3]_1^4 \qquad = [4 + 32 - \frac{64}{3}] - [1 + 2 - \frac{1}{3}]$

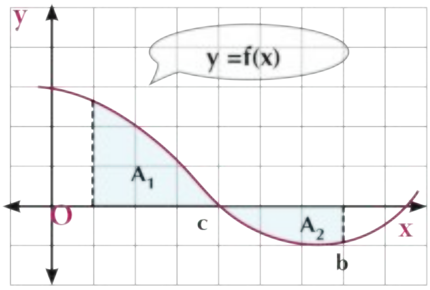
$= 36 - \frac{64}{3} - 3 + \frac{1}{3} = 12$ square unit

Try to solve

1. Find the area of the region bounded by the curve of function f, x-axis and the two striaight lines x = -1 and x = 2 where $f(x) = 3x^2 + 1$

Critical thinking

If the curve of the function f intersects x-axis at x = c where c ∈ [a, b], show how the area of the region on the x-axis and bounded by the curve of the function f and one of the two straight lines x = a and x = b (area A_1)can be calculated.



Notice that:

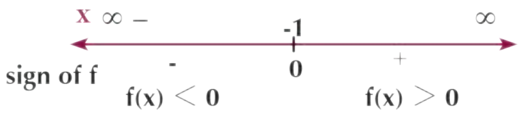
- To find the area , it is favorable to find the zeros of the function even the limits of the integral are given
- Studying the sign of the function on the subintervals if existed , then:
  - Positive i.e. f(x) > 0 on the interval [a, c] , then $A_1 = \int_a^c f(x)\,dx$
  - Negative i.e. f(x) < 0 on the interval [c, b], then $A_2 = |\int_c^b f(x)\,dx|$

Example The unction

2. Find the area of the region bounded by the curve of the function f: $f(x) = \sqrt[3]{2x + 2}$ and the straight line x = 3 and on the x-axis.

Solution

Find the zeroes of the function by putting f(x) = 0



$\therefore \sqrt[3]{2x + 2} = 0 \qquad$ i.e. : x = -1

$\therefore$ the required area $A = \int_{-1}^{3} f(x)\,dx$

$= \int_{-1}^{3} (2x + 2)^{\frac{1}{3}}\,dx = [\frac{3}{4 \times 2}(2x + 2)^{\frac{4}{3}}]_{-1}^{3}$

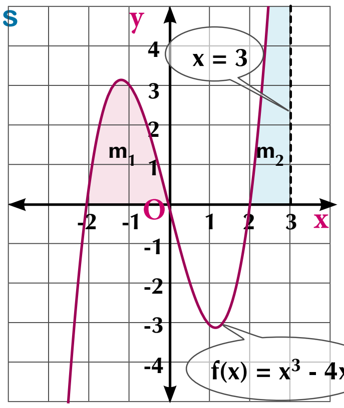
$= \frac{3}{8}[8^{\frac{4}{3}} - 0] = \frac{3}{8}(2^3)^{\frac{4}{3}} = 6$ square units

Try to solve

2 Find the area of the region above x-axis bounded by the curve of the function f: $f(x) = \frac{4x}{x^2 + 1}$ and the straight line x = 4 .

Example

The area bounded by a curve and x-axis



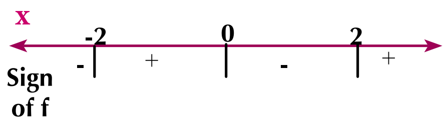
3 If f:]-∞, 3] → R where $f(x) = x^3 - 4x$, find the area of the region above x-axis bounded by the curve of the function and x-axis.

Solution

Find the intersection points of the curve of the function with x-axis (zeroes of the function)

$f(x) = x^3 - 4x = x(x^2 - 4) = x (x - 2) (x + 2)$

when f(x) = 0 $\therefore$ x = 0 or x = 2 or x = -2



By studying the sign of the function f, we find

$f(x) \geqslant 0$ on the interval [-2, 0] and on the interval [2, 3]

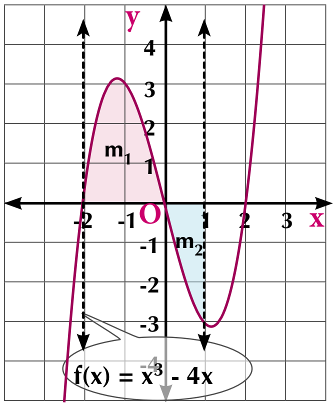
$\therefore$ Area $A = A_1 + A_2 = \int_{-2}^{0} (x^3 + 4x)\, dx + \int_{2}^{3} (x3 - 4x)\, dx$

$= \frac{1}{4} x^4 - 2x2]^0_{-2} + [\frac{1}{4} x^4 - 2x^2]^3_2$

$= (0 - (-4)) + (\frac{81}{4} - 18 - (-4))$

$= \frac{121}{4}$ square units

Important note



To identify the area bounded by the curve of the function, x-axis and the two straight lines x = -2 and x = 1 as shown in the graph opposite.

We find that:

$f(x) \geqslant 0$ when $x \in [-2, 0]$, $f(x) \leqslant 0$ when $x \in [0, 1]$

$\therefore$ Area A $= A_1 + A_2$

$= \int_{-2}^{0} (x^3 - 4x)\, dx + \left| \int_{0}^{1} (x^3 - 4x)\, dx \right|$

$= [\frac{1}{4} x^4 - 2x^2]^0_{-2} + \left| \frac{1}{4} x^4 - 2x^2]^1_0 \right|$

$= 4 + \left|(\frac{1}{4} - 2) - 0\right| = 4 + \left|-\frac{7}{4}\right| = \frac{23}{4}$ square units.

Try to solve

3) Find the area of the plane region bounded by the curve $y = 3 + 2x - x^2$ and x-axis.

Critical thinking

Find the area of the plane region bounded by the curve $y = 3 + 2x - x^2$ and the two straight lines $x = -1$, $x = 4$ and $y = 0$

Example Architectural applications of the area

4) An architect has designed an arc -like enteryway of a hotel whose equation $y = -\frac{1}{2}(x - 1)(x - 7)$ where x in meters. How much does the glass cost if this entryway is covered by the glass which costs 1500 LE per square meter?

Solution

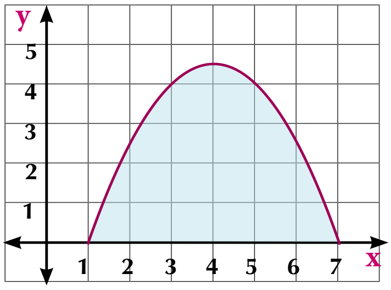
Modeling the problem:

The cost of the glass of the hotel entryway = area of the glass in square meters × cost of a square meter

Let the total cost be k and the area of the glass be A square meter

$\therefore \quad k = 1500\,A$ (1)

Finding the area of the glass:



Suppose the horizontal plane is the x-axis whose equation $y = 0$ and the equation of the arc of the hotel entryway $y = f(x)$ where:

$f(x) = -\frac{1}{2}(x - 1)(x - 7)$

$\therefore$ When $f(x) = 0$ then: $x = 1$ or $x = 7$

Then $f(x) \geqslant 0$ for each $x \in [1, 7]$

Area $A = \int_1^7 -\frac{1}{2}(x - 1)(x - 7)\,dx = \int_1^7 \left(-\frac{1}{2}x^2 + 4x - \frac{7}{2}\right)dx$

$= \left[-\frac{1}{6}x^3 + 2x^2 - \frac{7}{2}x\right]_1^7 = \frac{49}{3} - \left(-\frac{5}{3}\right) = 18$ (2) from (1), (2)

$\therefore k = 1500 \times 18 = 27000$

i.e.: The cost of covering the hotel entryway by glass equals 27000 LE

Try to solve

4) If the cost of a squared meter of granite to cover the floor of a hotel corridors is 400 LE and five corridors have been already covered with granite and the area of each is bounded by the curve of the function f and the two straight lines $x = 0$ and $y = 0$ where $f(x) = 12 - \frac{1}{3}x^2$, **find the cost of covering the five corridors**.

Second : Area of the plane region bounded by two curves

Definition

If f and g are two continuous functions on the interval [a, b], and $f(x) \geqslant g(x)$ for each $x \in [a, b]$, then the area of the region bounded by the two curves y = f(x) , y = g (x) and the two straight lines x = a and x = b is given by the relation

$A = \int_a^b [f(x) - g(x)]\, dx$

In the opposite figure , notice that:

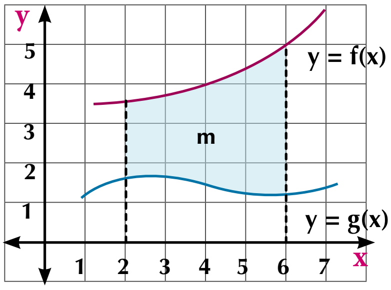
f and g are continuous on the interval [a, b]

$f(x) > g(x)$ for each $x \in [a, b]$

If the area between the curve of f(x) and x-axis = A_1

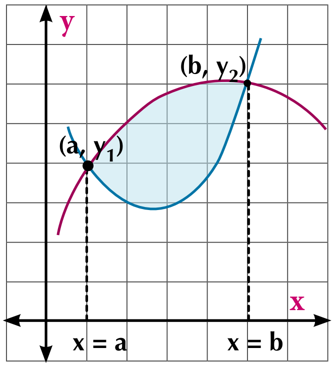
and the area between the curve g(x) and x-axis = A_2

then the area A between the curve of f(x) and g(x) = $A_1 - A_2$



i.e.: $A = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b [f(x) - g(x)]\, dx$

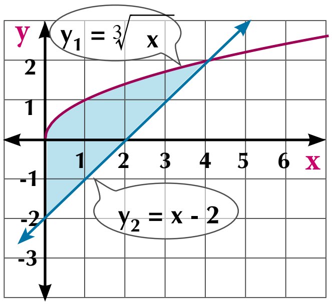
Important note : When a region gets bounded by intersected curves, then the limits of integral with respect to x are the x-coordinates of the intersection points.



Example Area of a region bounded by two intersected curves

(5) Find the area of the region bounded by the curve $y_1 = \sqrt{x}$, the straight line $y_2 = x - 2$ and y-axis

Solution



to find the x-coordinates of the intersection points, put $y_1 = y_2$

$x - 2 = \sqrt{x}$ by squaring both sides

$x^2 - 4x + 4 = x$ i.e: $x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$ $\therefore x = 1$ or $x = 4$

when x = 1 $y_1 = \sqrt{1} = 1,\ y_2 = 1 - 2 = -1$

i.e. $y_1 \neq y_2$

$\therefore$ when x = 1 there are not intersection points for the two curves

when x = 4 $y_1 = \sqrt{4} = 2,\ y_2 = 4 - 2 = 2$ $\therefore y_1 = y_2$

$\therefore$ the tow limits of the integral are x = 4 and x = 0 (y-axis) and the two curves of y_1 , y_2 are continuous on the interval [0, 4]

we take an arbitrary value belonging to the interval] 0, 4[let it be x = 2

when x = 2 $y_1 = \sqrt{2}$, $y_2 = 2 - 2 = 0$

i.e. $y_1 \geqslant y_2$ for each $x \in [0, 4]$

$\therefore$ Area of the region $= {}_0\int^4 (y_1 - y_2)\, d\,x = {}_0\int^4 (\sqrt{x} - x + 2)\, d\,x$

$= [\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{2} x^2 + 2x]^4_0 = \frac{2}{3} (2^2)^{\frac{3}{2}} - \frac{1}{2} (16) + 2\,(4) = \frac{16}{3} - 8 + 8 = \frac{16}{3}$ squared units

Try to solve

5) Find the area of the region bounded by the curve of the two functions f, and g where:

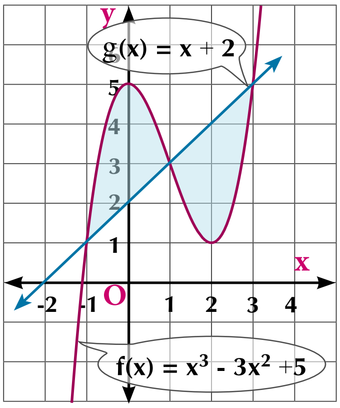
$f(x) = x^2 - 2$, $g(x) = 3 - (x + 1)^2$

Example

Plurality of regions bounded by two intersected curves

6) Find the area of the region bounded by the curve of the function f and the curve of the function g where

$f(x) = x^3 - 3x^2 + 5,\ g(x) = x + 2$



Solution:

➢ To find the x-coordinates of the intersection point:

put $f(x) = g(x)$

$x^3 - 3x^2 + 5 = x + 2$

$x^3\ 3\ _x^2 - x + 3 = 0$

$(x^3 - 3x^2) - (x - 3) = 0$

$x^2\,(x - 3) - (x - 3) = 0$

$(x - 3)\,(x - 1)\,(x + 1) = 0$

$\therefore x = 3$ or $x = 1$ or $x = -1$

➢ The integral is on the two intervals [-1, 1] and [1, 3] to find the required area which is two areas i.e.: $A = A_1 + A_2$

$A = |{}_{-1}\int^1 [f(x) - g(x)]\, d\,x| + |{}_1\int^3 [f(x) - g(x)]\, d\,x|$

$= |{}_{-1}\int^1 (x^3 - 3x^2 + 5 - x - 2)\, d\,x| + |{}_1\int^3 (x^3 - 3x^2 - x + 3)\, d\,x|$

$= |[\frac{1}{4}x^4 - x^3 - \frac{1}{2} x^2 + 3x]^1_{-1}| + |[\frac{1}{4} x^4 - x^3 - \frac{1}{2} x^2 + 3x]^3_1|$

$= |-2 + 6| + |26 - 30| = 4 + 4 = 8$ square units

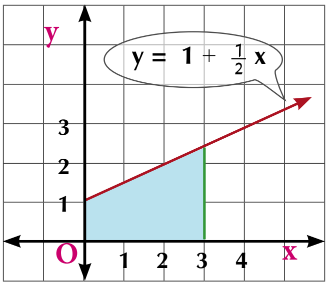
Try to solve

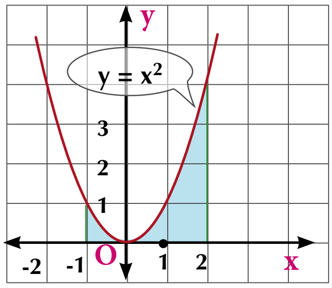
6) An advertising company produces a poster to market an item. If the poster is shaped as an area bounded by the curve of the two functions f and g where $f(x) = 2x^2$ and $g(x) = x^4 - 2x^2$ and x is approximated in decimeter, calculate the area needed of adhesive paper to produce 1000 posters for this item.

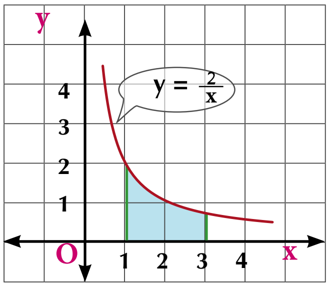
➢ Use a computer graphic program to graph this poster and search for the simplest method to find its area.

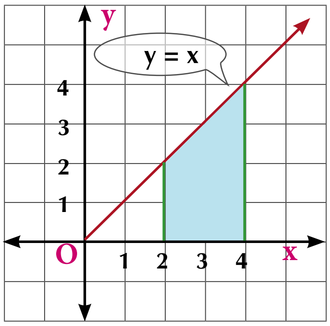
Exercises 4 - 4

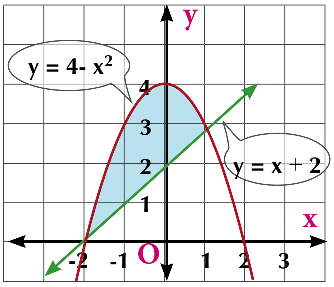
Write down the difinite integration which gives the colored area in rach of the following and calculate its value.

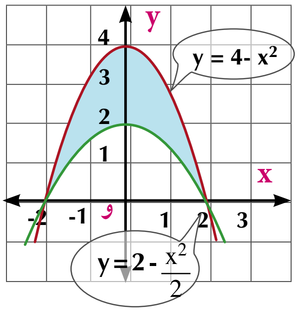
1


2


3


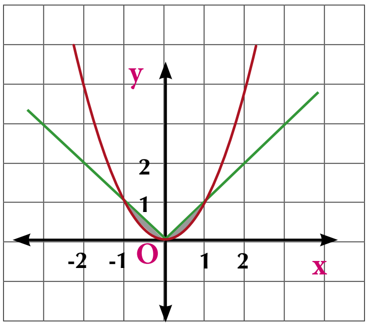
4


5


6


Choose the correct answer.

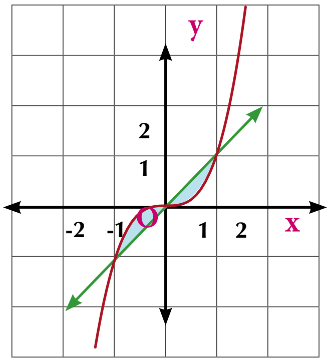
7. The area of the region bounded by the two curves $y = x^2$ and $y = |x|$ equals:



a) $2\int_{-1}^{0} (x^2 - x)\, dx$ b) $\int_{0}^{1} (x - x^2)\, dx$

c) $2\int_{0}^{1} (x - x^2)\, dx$ d) $\int_{-1}^{1} (x - x^2)\, dx$

8. The area of the region bounded by the curve $y = x^3$ and the straight line $y = x$, equals:



a) $\int_{-1}^{1} (x^3 - x)\, dx$ b) $2\int_{0}^{1} (x^3 - x)\, dx$

c) $\int_{0}^{1} (x - x^3)\, dx$ d) $2\int_{0}^{1} (x - x^3)\, dx$

9. The area of the region bounded by the straight lines $y = x$, $x = 2$ and $y = 0$: equals:

a) $\frac{1}{2}$ b) 1

c) 2 d) 4

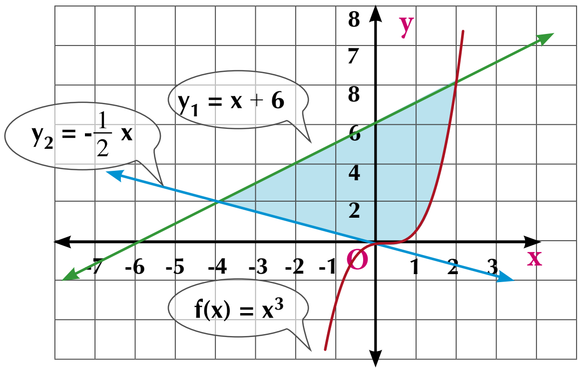
10. The area of the region bounded by the curve $y = x^3$ and the straight lines $y = 0$ and $x = 2$ equals

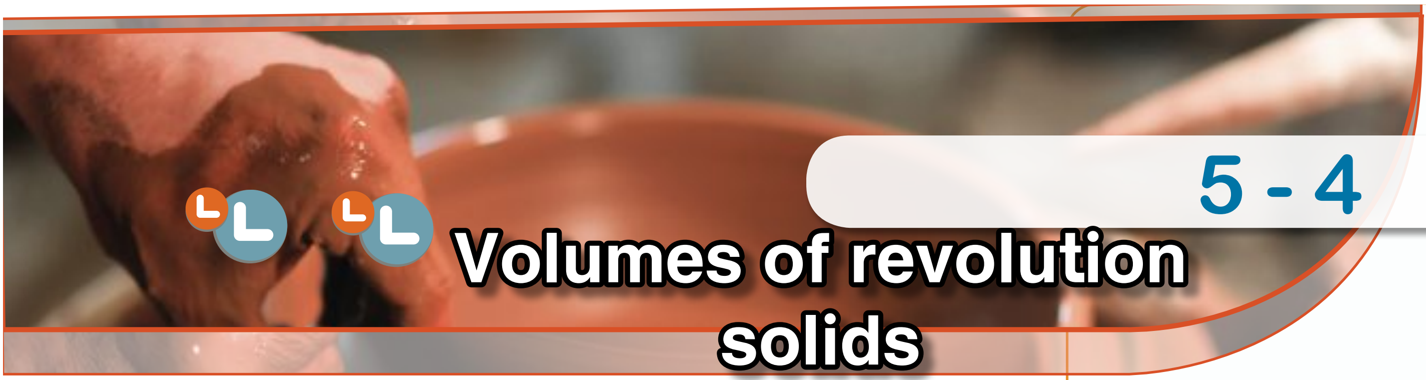
a) 8 b) 4 c) 2 d) 1

For each of the following, calculate the area of the plane region bounded by:

11. The curve $y = 5 - x^2$, x-axis and the two straight lines $x = -2$ and $x = 1$
12. The straight lines: $x + 2y = 9$, $x = 1$, $x = 3$, $y = 0$
13. The curve $y = \sqrt{x + 4}$ and the straight lines $x = 0$, $x = 5$, $y = 0$
14. The curve $y = 3 - 2x - x^2$ and x-axis
15. The curve $y = \frac{4}{x^2}$ and the straight lines $x = 1$, $x = 4$, $y = 0$
16. The curve of the function f: $f(x) = (3 - x)(x - 1)^2$ and the two coordinate axes where $f(x) \geqslant 0$
17. The curve of the function f: $f(x) = (x - 1)(x - 2)(x - 3)$ and the two straight lines $x = 4$, $y = 0$ where $f(x) \geqslant 0$
18. The curves of the two functions f and g where $f(x) = 2x^2$, $g(x) = 2x + 4$
19. Use the definite integration to prove that the area of the triangle whose base length equals a and its height equals b is $\frac{1}{2}$ ab
20. **Creative thinking:** in the opposite figure , find the area of the region bounded by the curve of the function f and the two straight lines y_1 and y_2 where:

$f(x) = x^3$, $y_1 = x + 6$, $y_2 = -\frac{1}{2}x$





5 - 4

Volumes of revolution solids

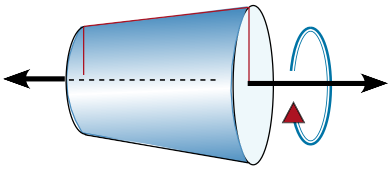
Think and discuss

Have you ever watched the pottery maker while he changes the dust into cooking utensils by blending the Aswani mud with water, cutting it and putting it around an axis of revolution. The pottery maker can shape this mud using his fingers and tools to produce very attractive solids. What are these solids named?

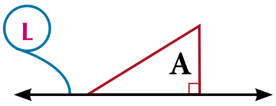
- Plastic containers for packing soft drinks, juice and oil are designed in different volumes and in several capacities. How can their volumes and capacities be calculated as they are being manufactured?

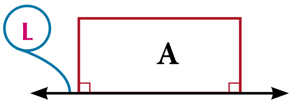
Solid of Revolution

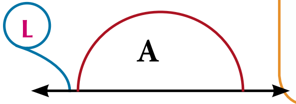
The solid of revolution is generated due to revolving a plane area a complete revolution about a fixed straight line in its plane called «axis of revolution».

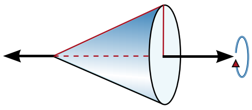


The following figures shows some examples for solids of revolution drawn by the area A as they revalve a complete revolution about the straight line **L**

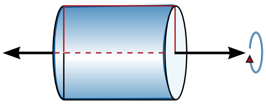




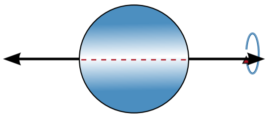




Right cone



Right cylinder



sphere

You will learn

- Identify the volume as a definite integration.
- Use the definite integration to find the volumes.
- Find a revolution volume generated by revolving a region bounded by two curves

Key terms

- Axis of Revolution
- Solid of Revolution

Materials

- Scientific calculator.
- Computer graphics.

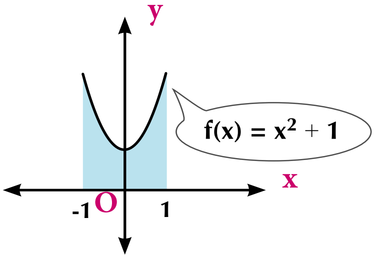
Volumes of revolution

First: the volume of the solid generated by revolving a plane region about an axis.

Theorem If f is a continuous function on the interval [a , b] , $f(x) \geqslant 0$ for $x \in [a , b]$, then the solid generated by revolving the area bounded by the curve $y = f(x)$, x-axis and the two straight lines $x = a$ and $x = b$ a complete revolution about x-axis is: $V = \pi \int_a^b [f(x)]^2 \, dx$

Example — Revolution about x-axis

1. Find the volume of the solid generated by revolving the plane region bounded by the curve of the function f , x-axis and the two straight lines $x = -1$ and $x = 1$ a complete revolution about x-axis known that $f(x) = x^2 + 1$

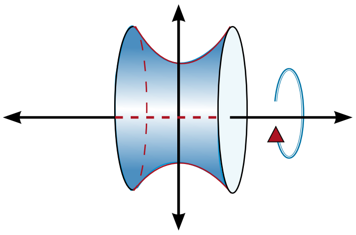


Solution:

The function f is polynomial and continuous on the interval [-1, 1],

$f(x) \geqslant 0$ for each $x \in [a, b]$

Let the volume of the solid generated by revolving = V



$$\therefore V = \pi \int_{-1}^{1} (x^2 + 1)^2 \, dx$$

$$= \pi \int_{-1}^{1} (x^4 + 2x^2 + 1) \, dx$$

$$= \pi \left[\frac{1}{5} x^5 + \frac{2}{3} x^3 + x\right]_{-1}^{1} = \frac{56}{15} \pi \text{ cubic unit}$$

Try to solve

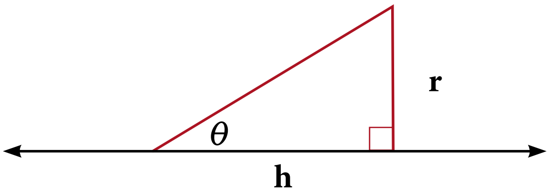
1. Find the volume of the solid generated by revolving the plane region bounded by the curve of function f , x-axis and the two straight line $x = 0$ and $x = 3$ a complete revolution about x-axis given that $f(x) = x$. what is the name of the generated solid? Show how you can check your answer geometrically.

Example — Applications of volumes

2. Use the integration to prove that the volume of the right circular cone equals $\frac{\pi}{3} r^2 h$ where r is the radius length of its base and h is its height.

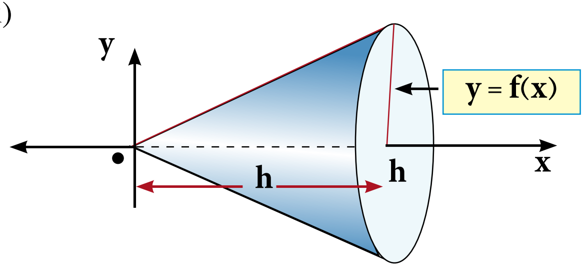
Solution:

The right circular cone is generated by revolving the right - angled triangle such that one leg of the right angle lies on the x-axis a complete revolution about x-axis.



Find the relation between x and y = f(x)

$\tan\theta = \frac{y}{x}$ (1) $\therefore y = x\tan\theta = f(x)$



$\because V = \pi \int_a^b [f(x)^2]\, dx = \pi \int_0^h x^2 \tan^2\theta \, dx$

$= [\frac{\pi}{3} x^3 \tan^2\theta]_0^h = \frac{\pi}{3} h^3 \tan^2\theta$ (2)

from (1) $\tan\theta = \frac{y}{x} = \frac{r}{h}$

$\therefore \tan^2\theta = \frac{r^2}{h^2}$

By substituting in (2) $\therefore V = \frac{\pi}{3} \frac{r^2}{h^2} \times h^3 = \frac{\pi}{3} r^2 h$

Try to solve

(2) Use the integration to prove that:

a) Volume of sphere = $\frac{4}{3}\pi r^3$
(r is the radius length of the sphere)

b) Volume of the right circular cylinder = $\pi r^2 h$
(r is the radius length of the cylinder base and h is its height)

Remeber

The equation of the circle whose center is the origin point (0, 0) and its radius length (r) is:
$x^2 + y^2 = r^2$

Example

Revolution about x-axis

(3) Find the volume of the solid generated by revolving the region bounded by the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and x-axis where a and b are two constants a complete revolution about x-axis.

Solution:

$\because$ Revolution around x-axis $\therefore y^2 = b^2 (1 - \frac{x^2}{a^2})$

Limits of integration:

y = 0 $\therefore x^2 = a^2$ **i.e** x = - a , x = a

$V = \pi \int_{-a}^{a} b^2 (1 - \frac{x^2}{a^2})\, dx = 2\pi b^2 \int_0^a (1 - \frac{x^2}{a^2})\, dx$ **(why?)**

$= 2\pi b^2 [x - \frac{x^3}{3a^2}]_0^a = 2\pi b^2 [a - \frac{1}{3} a]$

$= \frac{4}{3}\pi b^2 a$ cubic units.

Try to solve

(3) Find the volume of the solid generated by revolving the region bounded by the curve $y = 2x - x^2$ and x-axis a complete revolution about x-axis.

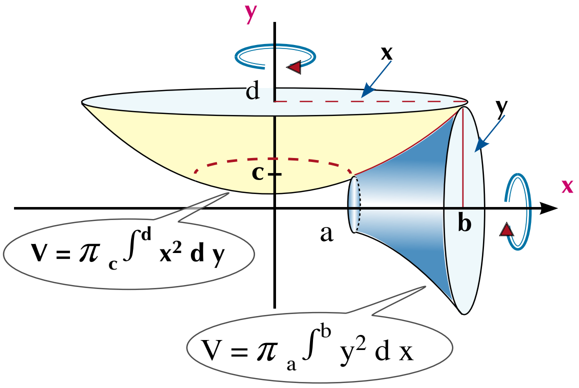
Important note

If revolving the plane region is about x-axis and the two straight lines x = a , x = b **then:**

$V = \pi \int_a^b y^2 \, dx$

If revolving the plane region is about y-axis and the two straight lines y = c , y = d **then:**

$V = \pi \int_c^d x^2 \, dy$



Example Revolution about y- axis

(4) Find the volume of the solid generated by revolving the region bounded by the curve $y = x^2 + 1$ and y-axis and the straight line y = 5 complete revolution about y-axis.

Solution:

$\because y = x^2 + 1$ and the revolution is about y-axis

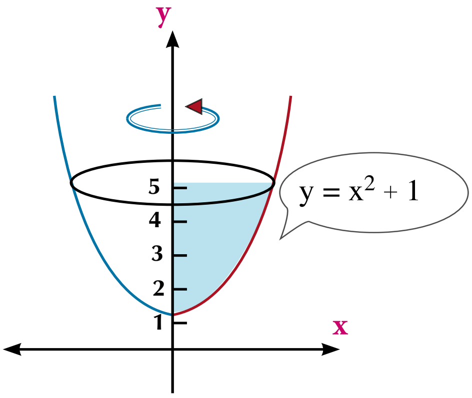
$\therefore x^2 = y - 1$

When x = 0 y = 1

limits of integral y = 1 , y = 5

$V = \pi \int_c^d x^2 \, dy = \pi \int_1^5 (y - 1) \, dy$

$= [\frac{\pi}{2} (y - 1)^2]_1^5 = \frac{\pi}{2}(16 - 0) = 8\pi$ cubic unit



Try to solve

(4) Find the volume of the solid generated by revolving the region bounded by the curve $y = x^2$, y-axis and the two straight lines y = 0 and y = 6 complete revolutions about y-axis.

Second: Volume of the soild generated by revolving a region bounded by two curves

Theorem

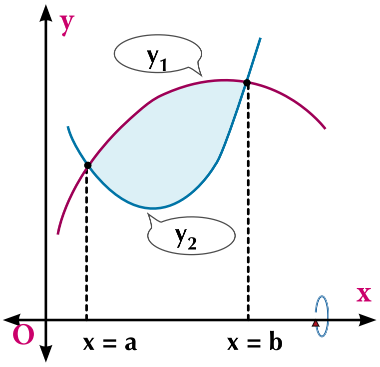
If f and g are two continuous functions on the interval [a, b] , f(x) ⩾ 0 , g (x) ⩾ 0 for each x ∈ [a, b], then the volume of the solid of revolution V generated by revolving the region bounded by the two curves and the two straight lines x = a , x = b a complete revolution about x-axis is:

$$V = \pi \int_a^b [[f(x)]^2 - [g(x)]^2] \, dx$$

Note that

1. If the region bound by the two intersected curves $y_1 = f(x)$, $y_2 = g(x)$ where $y_1 \geqslant y_2$ for each $x \in [a, b]$

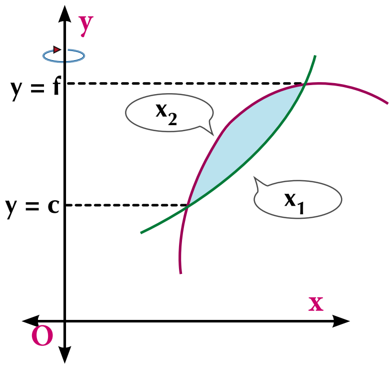
The colored area in the opposite figure revolves a complete revolution about x-axis, then the two x-coordinates of the two intersection points of the two curves are the limits of integration a and b where $a < b$ and the volume of the generated solid is:



$$V = \pi \int_a^b [(y_1^2 - y_2^2) \, dx$$

i.e: $V = \pi \int_a^b y_1^2 \, dx - \pi \int_a^b y_2^2 \, dx$

2. If the region bounded by the two intersected curves $x_1 = f(y)$ and $x_2 = g(y)$ where $x_1 \geqslant x_2$ for each $y \in [c, d]$ revolves a complete revolution about y-axis, then the y-coordinates of their two intersection points are the Limits of the integration c and d where $c < d$ and



$$V = \pi \int_c^d [(x_1^2 - x_2^2) \, dy$$

i.e: $V = \pi \int_c^d x_1^2 \, dy - \pi \int_c^d x_2^2 \, dy$

Example Revolving a region bounded by two curves about x-axis

(5) Find the volume of the solid generated by revolving the region bounded by the curve $y = x^2$ and the straight line $y=2x$ a complete revolution about x-axis.

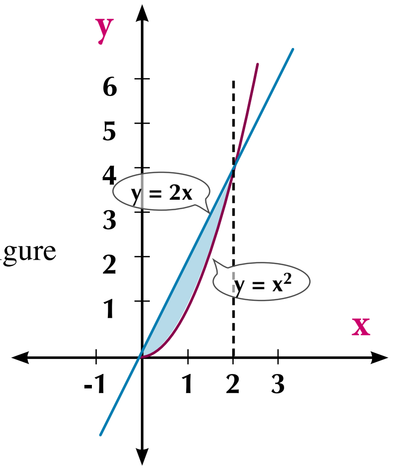
Solution:

Let $y_1 = x^2$, $y_2 = 2x$

to find the intersection points, put $y_1 = y_2$

$\therefore \; x = 0$ or $x = 2$

$\because \; y_2 \geqslant y_1$ for each $x \in [0, 2]$ as shown in the opposite figure



$\therefore \; V = \pi \int_a^b (y_2^2 - y_1^2) \, dx$

$= \pi \int_0^2 (2x)^2 - (x^2)^2 \, dx$

$\therefore \; V = \pi \int_0^2 (4x^2 - x^4) \, dx$

$= \pi \left[\frac{4}{3} x^3 - \frac{1}{5} x^5\right]_0^2$

$= \pi \left(\frac{32}{3} - \frac{32}{5}\right) = 32\pi \times \frac{2}{15} = \frac{64\pi}{15}$ cubic units

Try to solve

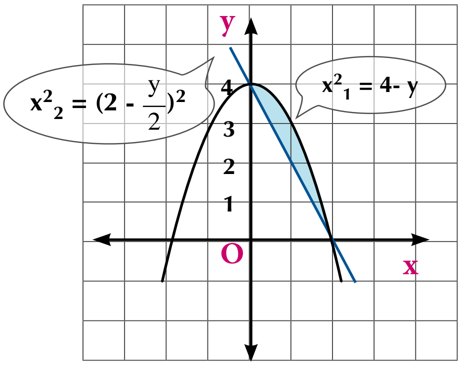
(5) Find the volume of the soild generated by revolving the region bounded by the two curves $y = \sqrt{x}$ and $y = x^2$ a complete revolution about x-axis.

Example

Revolving a region bounded by two curves about y-axis

(6) Find the volume of the solid generated by revolving a region bounded by the curve $y = 4 - x^2$ and the straight line $2x + y = 4$ a complete revolution.

Solution:



$\because$ Revolving about y-axis

$\therefore x^2_1 = 4 - y$, $x^2_2 = (2 - \frac{y}{2})^2$

At the intersection points $x_1 = x_2$

$4 - y = (2 - \frac{y}{2})^2$ $\qquad \therefore \frac{y^2}{4} - y = 0$

$y (y - 4) = 0$ $\qquad y = 0 , y = 4$

and $x^2_1 > x^2_2$ $\qquad$ for each $x \in [0, 4]$

$$V = \pi \int_c^d (x^2_1 - x_2^2)\, d y = \pi \int_0^4 [(4 - y) - (2 + \frac{y}{2})^2]\, dy$$

$$= \pi \int_0^4 (y - \frac{y^2}{4})\, d y = \pi \left[\frac{1}{2} y^2 - \frac{y^3}{12}\right]_0^4$$

$$= \pi [8 - \frac{16}{3}] = \frac{8}{3} \pi \text{ cubic units}$$

Try to solve

(6) Find the volume of the solid generated by revolving a region bounded by the two curves $2y = x^2$ and $y = \sqrt{x}$ a complete revolution about y-axis.

Exercises 4 - 5

Choose the correct answer:

(1) The volume of the solid generated by revolving a region bound by the curve $y = 2\sqrt{x}$, $y = 0$ and $x = 1$ a complete revolution about x-axis equals.

(a) $- \pi$ $\qquad$ (b) 0 $\qquad$ (c) π $\qquad$ (d) 2π

(2) The volume of the solid generated by revolving the region bounded by the curve $y = \frac{1}{x}$ $y = 1$ and $y = 2$ and the y-axis a complete revolution about y-axis equals.

(a) $\frac{\pi}{4}$ $\qquad$ (b) $\frac{\pi}{2}$ $\qquad$ (c) π $\qquad$ (d) 2π

3. The volume of the solid generated by revolving the region bounded by the curve $y = x^2$ and the straight line $y = 1$ a complete revolution about y-axis equals:

   a) π b) $\frac{1}{2}\pi$ c) $\frac{1}{4}\pi$ d) 2π

4. $\pi \int_{-2}^{2} (4 - x^2)\, dx$ is the volume of

   a) A sphere whose radius length is 4 units
   b) A right circular cone whose height is 4 units
   c) A sphere whose radius length is 2 units
   d) A right Circular cylinder whose height is 4 units

Find the volume of the solid generated by revolving the region bounded by the given curves and straight lines a complete revolution about x-axis for each of the following:

5. $y = x$, $x = 3$, $y = 0$
6. $y = 3 - x$, $x = 0$, $y = 0$
7. $y = \frac{1}{x}$, $x = 1$, $x = 4$, $y = 0$
8. $y = |x|$, $x = -2$, $x = 4$, $y = 0$

Find the volume of the solid generated by revolving the region bounded by the given curves and straight lines a complete revolution about y-axis for each of the following:

9. $y = x$, $y = 1$, $x = 0$
10. $y = x^2$, $x = 0$, $y = 0$, $y = 8$
11. $y = 4 - x^2$, $x = 0$, $y = 0$
12. $y = x^3$, $x = 0$, $y = 0$, $y = 8$
13. $x + 2y = 0$, $x = 0$, $y = 0$, $y = 3$

Answer the following:

14. Find the volume of the solid generated by revolving the curve $y^2 = 4 - x$ and the straight line $x = 0$ when this region revolves a complete revolution.

    First: about x-axis **Second**: about y-axis

15. Find the volume of the solid generated by revolving the curve $y = \frac{4}{x}$ and the straight line $x + y = 5$ a complete revolution about x-axis.

16. **Critical thinking:** if points A(-2, 0) , B(1, 5) and C(4, 0) are the vertices of the triangle ABC, find the volume of the solid generated by revolving the triangle ABC a complete revoution about x-axis.

Unit Summary

Differential of function: If the function f is differentiable on an open interval containing x , y = f(x), then:

$dy = f'(x)\,dx$ where d y is differential of y and d x is differential of x.

Integration by substitution: It is one of the methods to find the integration of the product of two functions. We can use it to change the given integration into know measure integration. If z = g (x) is a differentiable function, then:

$$\int f(g(x))\, g'(x)\, dx = \int f(z)\, dz$$

Integration by parts: it is one of the methods to find the integration of the product of two functions each of them is not a derivative to the other. if y and z are two differentiable functions on the interval I **Then**: $\int y\, dz = y z - \int z\, dy$

Table of fundamental integrations (standard)	
$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \qquad n \in \mathbb{R} - \{1\}$	$\int \sec x \tan x\, dx = \sec x + c$ $x \neq \frac{2^n+1}{2}\pi$, n ∈ Y
$\int \sin x\, dx = -\cos x + c$	$\int \csc x \cot x\, dx = -\csc x + c$ $x \neq n\pi$, n ∈ Y
$\int \cos x\, dx = -\sin x + c$	$\int e^x\, dx = e^x + c$
$\int \sec^2 x\, dx = \tan x + c$ $x \neq \frac{2n+1}{2}\pi$, $n \in Z$	$\int a^x\, dx = \frac{1}{\ln a} a^x + c \qquad a > 0 , a \neq 1$
$\int \csc^2 x\, dx = -\cot x + c$ $x \neq n\pi$, $n \in Z$	$\int \frac{1}{x}\, dx = \ln \lvert x \rvert + c \qquad x \neq$ zero

- Adding a constant (b) to the independent variable does not affect the formula of integration.
- As multiplying the variable x by the coefficient (a), the integration keeps its previous formula but it divides this coefficient.

Definite integration:

Theorem:

If the function f is continuous on the interval [a , b] and F is any antiderivative to the function f on the same interval, then $\int_a^b f(x)\, dx = F(b) - F(a)$

Properties of the definite integral:

① $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$ ② $\int_a^a f(x)\, dx = 0$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for each $c \in [a, b]$
4. $\int_{-a}^{a} f(x)\,dx = 0$ (f is an add function)
5. $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ (f is an even function)

- The area of a region bounded by the curve of the continuous function f on the interval [a, b] and the two straight lines

  $x = a$ and $x = b$ where $f(x) \geqslant 0$ is $A = \int_a^b f(x)\,dx$

- The area of a region bounded by the curve of the two continuities functions f and g on the interval [a , b] and the two straight lines

  $x = a$ and $x = b$ where $f(x) \geqslant g(x) > 0$ is: $A = \int_a^b [f(x) - f(g)]\,dx$

Volumes of revolution:

The solid of revolution is generated by revolving a plane region a compete revolution about a straight line is called the axis of revolution.

- The volume of a solid generated by revolving a region bounded by the curve of the continuous function f on the interval [a, b] x-axis and the two straight lines $x = a$ and $x = b$ a complete revolution about x-axis where $f(x) \geqslant 0$ is

  $V = \pi \int_a^b [f(x)]^2\,dx$

- The volume of a solid generated by revolving the region bounded by the two continuous functions f and g on the interval [a, b] and the two straight lines $x = a$ and $x = b$ a complete revolution about x-axis where $f(x) \geqslant g(x) > 0$ is :

  $V = \pi \int_a^b [(f(x))^2 - (g(x))^2]\,dx$

General Exercises

Choose the correct answer:

1. If $f(x) = \int (3x^2 - 5)\, dx$ and $f(2) = 3$, then $f(-2)$
   - a) -6
   - b) -3
   - c) 7
   - d) 12

2. If $\frac{dy}{dx} = x + \frac{1}{x}$, $y = \frac{1}{2}$ when $x = 1$, and when $x = e$, then y equals:
   - a) $e^2 - e$
   - b) $\frac{e^2 - 1}{2}$
   - c) $\frac{e^2 + 1}{4}$
   - d) $\frac{e^2}{2} + 1$

3. $\int \tan^2 x\, dx$ equals
   - a) $\tan x - x + c$
   - b) $\tan x + x + c$
   - c) $\sec^4 x + c$
   - d) $\frac{1}{3}\tan^3 x + c$

4. If $\int_2^5 f(x)\, dx = 4$, then $\int_2^5 [3 f(x) - 1]\, dx$ equals
   - a) 9
   - b) 11
   - c) 12
   - d) -8

5. The volume of the solid generated by revolving the region bounded by the curve $f(x) = x^2$ and x-axis and the two straight lines $x = -2$, $x = 2$ one revolution about x-axis equals
   - a) $\frac{16\pi}{5}$
   - b) $\frac{32\pi}{5}$
   - c) $\frac{64\pi}{5}$
   - d) 4π

Find the differential for each of:

6. $y = 3x^2 - 1$
7. $y = \sqrt{(2x + 3)^5}$
8. $y = (x + \frac{1}{x})^3$
9. $y = e^{-5x}$
10. $y = e^{x^2 + 3x}$
11. $y = e^{\cos 2x}$
12. $z = \ln (x^2 + 1)$
13. $z = (\ln \frac{10}{x})^2$
14. $z = \ln \cot (3x - 1)$

Express each of the following using one integration

15. $\int_0^5 x^2\, dx + \int_{-2}^0 x^2\, dx$
16. $\int_{-4}^{-1} \sqrt{x + 6}\, dx - \int_4^{-1} \sqrt{x + 6}\, dx$
17. $\int_2^6 k\, dx + \int_3^2 k\, dx$
18. $\int_0^4 2x^2\, dx + \int_4^5 x^2\, dx$

Answer the following:

19. If $\int_2^4 f(x)\, dx = 5$, $\int_2^4 g(x)\, dx = -3$, find:
   - a) $\int_2^4 3f(x)\, dx$
   - b) $\int_4^2 2\, g(x)\, dx$
   - c) $\int_2^4 [3f(x) - 2\, g(x)]\, dx$
   - d) $\int_2^4 [3g(x) + 4x]\, dx$

20. If $f''(x) = 6x - 4$, $f'(1) = 2$, $f(0) = -4$, find $f(x)$

Find the following definite integral:

21. $\int_2^4 (x^2 + \frac{1}{x^2})\, dx$
22. $\int_1^3 \frac{6 - x^4}{2x^2}\, dx$
23. $\int_1^4 \frac{x - 1}{\sqrt{x} - 1}\, dx$
24. $\int_{-1}^1 \frac{x + 8}{\sqrt[3]{x} + 2}\, dx$
25. $\int_{-4}^4 \sqrt{16 - x^2}\, dx$
26. $\int_0^2 \sqrt{4 - x^2}\, dx$

Use the proper substitution to find the following integral:

27) $\int x\,(x-5)^3\,dx$

28) $\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})^3}$

29) $\int x \sec^2(x^2+2)\,dx$

30) $\int \sec^4 x \tan x\,dx$

31) $\int x\, e^{x^2+3}\,dx$

32) $\int \frac{e^{2x}+x}{e^{2x}+x^2}\,dx$

Use the proper parts to find the following integral:

33) $\int 3x\sqrt{2x+3}\,dx$

34) $\int x^4 \ln x\,dx$

35) $\int_1^2 4x\, e^{2x}\,dx$

Answer the following:

36) Find the area of the region bounded by the two curves $y = 7 + 2x - x^2$, $y = (x - 1)^2$

37) Find the area of the region bounded by the two curves $y = 9 - x^2$, $y = x^2 + 1$ and the two straight lines $x = 0$, $x = 3$

38) Find the area of the region bounded by the two curves $y = 9 - x^2$, $y = x^2 + 1$ and x-axis and the two straight lines $x = 0$, $x = 3$

39) Find the volume of the solid generated by revolving the region bounded by the curve $y = \frac{1}{x-1}$ the two straight line $x = 2$ and $x = 4$ and x-axis a compete revolution about.

a) x - axis

b) y-axis

40) Find the volume of the solid generated by revolving the region bounded by the curve $y = \sqrt{2x}$, x-axis and the tangent to the curve at point (2, 2) lying on it when this point revolves a complete revolutions about x-axis.

For more activities and exercises, visit the next site: www.sec3mathematics.com.eg

Accumulative test

Choose the correct answer:

1. $\int (4 - \csc x \cot x)\, dx$ equals:

   a) $4x - \csc x + c$ b) $4x + \csc x + c$
   c) $4x - \cot x + c$ d) $4x + \cot x + c$

2. $\int \frac{e^x}{e^x - 3}\, dx$ equals:

   a) $-\frac{1}{2}(e^x - 3) + c$ b) $-\ln|e^x - 3| + c$
   c) $\ln|e^x - 3| + c$ d) $\frac{1}{2}\ln|e^x - 3| + c$

3. $\int_0^2 (2 - |x|)\, dx$ equals:

   a) 4 b) 2 c) 1 d) zero

4. The area of the region bounded by the curve $y = \sqrt{4 - x^2}$ and x-axis approximated in squared units equals: ....

   a) 2 b) 4 c) 2π d) 4π

5. If $\int_1^3 f(x)\, dx = 5$, $\int_3^1 g(x) = 7$, then $\int_1^3 (4 f(x) + g(x) + 3)\, dx$ equals:

   a) -12 b) -7 c) 12 d) 19

6. The volume of the solid generated by revolving the region bounded by the curve $y = \frac{2}{x}$ and the straight lines $x = 1$, $x = 4$ and $y = 0$ a complete revolution about x-axis approximated in cubic units equal: ....

   a) $\frac{\pi}{3}$ b) $\frac{\pi}{2}$ c) 2π d) 3π

Answer the following:

7. Find the following integrations:

   a) $\int \frac{x + 3}{\sqrt{x^2 + 6x}}\, dx$ b) $\int x^2 \sqrt{x - 3}\, dx$

8. If $f(x) = \int (x + 1)(2x^2 + 4x - 1)\, dx$, $f(-2) = 1$, **find f(3)**

9. Find the following integrations:

   a) $\int \tan x \sec^3 x\, dx$ b) $\int \sqrt{1 + \sin x}\, \cos x\, dx$

Accumulative test

10 Find the following integrations:

a $\int x^2 \ln x^2 \, d x$ **b** $\int x e^{2x^2 - 1} \, d x$

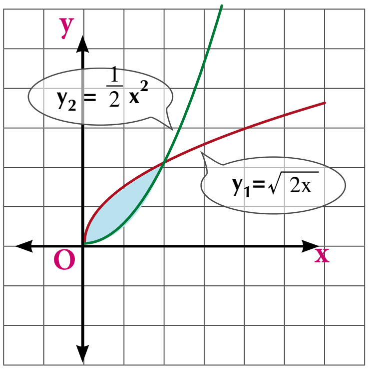
11 Find the value for each of the following:

a $\int_{-3}^{3} (2x^5 - 3 \sin \frac{\pi}{6} x) \, d x$ **b** $\int_{-2}^{2} x^2 (3 - 2 |x|) \, d x$

12 If $\int_{2}^{5} f(x) \, d x = 8$, $\int_{5}^{2} g(x) \, d x = 3$, calculate the value of $\int_{2}^{5} (f(x) + 2g(x) - 5) \, d x$

13 Find in square units the area of the region bounded by the curve of $f(x) = (x - 2)^3$ and the x-axis in the interval [2, 4]

14 The opposite figure shows the region bounded by the two curves $y_1 = \sqrt{2x}$ and $y_2 = \frac{1}{2} x^2$, find:



a Area of the region bounded by y_1 and y_2

b The volume of the solid generated by revolving the region bounded by the two curves y_1 and y_2 a complete revolution about x-axis.

If you cannot answer any of these questions, you can use the next table:

Question no	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Back to	2	1	3	4	3	5	1	skills	2	1	3	3	4	5

General test

Test 1

First: Answer the following question

1 Choose the correct answer

1- Which of the following functions satisfies the realation $\frac{d^3y}{dx^3} = y$

a) $y = \frac{1}{12}(x+1)^4$ b) $y = \sin x$

c) $y = e^{x}$ d) $y = \frac{x}{x-1}$

2- If the radius length of a circle increases at a rate $\frac{1}{\pi}$ cm/s, the circumference of the circle increases at a rate of .... cm/sec

a) $\frac{2}{\pi}$ b) 2 c) π d) 2π

3- The curve of the function f where $f(x) = x^3 - 3x^2 + 2$ is convex upwards when $x \in$

a) $]-\infty, 0[$ b) $]-\infty, 1[$ c) $]1, 3[$ d) $]1, \infty[$

4- $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin x + \cos x)\, dx$ equals

a) 4 b) 2 c) zero d) π

5- If f is a continuous function on R, $\int_3^5 2 f(x)\, dx = 8$, $\int_3^4 3 f(x)\, dx = 9$, then $\int_4^5 5 f(x)\, dx$ equals

a) zero b) 1 c) 3 d) 5

6- The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and x-axis approximated in square units equals:

a) 16π b) 12π c) 8π d) 4π

Second: Answer three questions only of the following

2 a) Find: $\int \sin x \cos^3 x\, dx$ b) If $e^{xy} - x^2 + y^3 = 0$, find $\frac{dy}{dx}$ when $x = 0$

3 a) Find the equation of the tangent to the curve $x^2 - 3xy - y^2 + 3 = 0$ at point (-1, 4)

b) The lengths of the legs of the right angle a right - angled triangle at a moment, are 6cm and 30cm. If the length of the first leg increases at a rate of $\frac{1}{3}$ cm/min and the length of the second leg decreases at a rate of 1cm/min, find:

1- The rate of increase in the area of the triangle after 3 minutes

2- The time at which the increase of the area of the triangle stops.

4 a) Determine the increasing and decreasing intervals to the function f where $f(x) = x + 2\sin x,\ 0 < x < 2\pi$

b) A rectangle is drawn such that two adjacent vertices of the rectangle lie on the curve $y = x^2 - 12$ and the other two vertices lie on the curve $y = 12 - x^2$, find the maximum area of this rectangle.

5 **a** Find the volume of the solid generated by revolving the region bounded by the two curves $y = \frac{4}{x}$ and $y = (x - 3)^2$ a complete revolution about x-axis

b Sketch the curve of the function f which satisfies the following properties:

$f(1) = f(5) = 0$, $f(2) = -3$ $\qquad f'(x) < 0$ for each $x \neq 2$

$f'(x) < 0$ for each $x < 2$, $\qquad f'(x) > 0$ for each $x > 2$

Test 2

First: Answer the following question:

1 **Choose the correct answer:**

1- The equation of the tangent to the curve of the function f where $f(x) = e^{2x+1}$ at point $(\frac{-1}{2}, 1)$ is:

a $2y = x + 1$ b $y = 2x + 2$ c $y = 2x - 3$ d $2y = 3x + 1$

2- If $y = 4n^3 + 4$, $z = 3n^2 - 2$, then the rate of change of z with respect to y equals:

a $2n$ b 2 c $\frac{1}{2n}$ d 4

3- The maximum value of the expression $8x - x^2$ where $x \in R$ is

a 8 b 16 c 32 d 64

4- If the slope of the tangent to the curve of the function f at any point on it equals $\frac{1}{x - 2}$ and the curve passes through point (3 , 0), then $f(e^2 + 2)$ equals

a 2 b 3 c ln 2 d ln 3

5- If f is a continuous function on R , $\int_1^2 f(x)\, dx = 9$ $\int_6^2 f(x)\, dx = -7$, then $\int_1^6 f(x)\, dx$ equals:

a 2 b 8 c 16 d -63

6- The volume of the solid generated by revolving the region bounded by the curve $y = \sqrt{x + 1}$ and the straight lines $y = 0$, $x = -1$ and $x = 1$ equals

a π b $\frac{3\pi}{2}$ c 2π d $\frac{5\pi}{2}$

Second: Answer three question only of the following :

2 Find

a $\int x(2x - 1)^3\, dx$, $\int x e^{-2x}\, dx$

b Find the rate of change for $\sqrt{16 + x^2}$ with respect to $\frac{x}{x - 2}$ when $x = -3$

(3) **a** If $x \cos y + y \cos x = 1$, find $\frac{dy}{dx}$

b Find the absolute extrema values of the function f in the interval $[-1, 1]$ where $f(x) = 2x^3 + 6x^2 + 5$

(4) **a** If $f(x) = \begin{cases} 2x + x^2 & \text{when } x < 0 \\ 2x - x^2 & \text{when } x \geq 0 \end{cases}$, find:

1- The local maximum and minimum values of function f

2- $\int_{-1}^{3} f(x)\,dx$

b the volume of a cube increases regularly such that it keeps its shape at a rate of 27 cm^3/min, find the increase of the area of its faces at the moment which its edge length is 3 cm.

(5) **a** Find the area of the region bounded by the two curves $y = x^2$ and $y = 6x - x^2$ in square units

b If the function f where $f(x) = x^3 + a x^2 + b x$ has an inflection point at (2, 2), find the two values of the two constants a and b, then sketch the curve of the function.

Test 3

First: Answer the following question:

(1) **Choose the correct answer**

1- The slope of the tangent to the curve of the circle $x^2 + y^2 = 25$ when $x = 3$ equals

a $\frac{-4}{3}$ **b** $\frac{-3}{4}$ **c** $\frac{5}{12}$ **d** $\frac{4}{3}$

2- If $f(x) = \frac{x}{x-2}$, then $f'''(3)$ equals

a -36 **b** -12 **c** 6 **d** 4

3- If $\frac{dy}{dx} = \csc^2 x$, $y = 2$ and $x = \frac{\pi}{4}$, then y equals

a $-(2 + \cot x)$ **b** $-(3 + \cot x)$ **c** $2 - \cot x$ **d** $3 - \cot x$

4- If $\int_{2}^{4} f(x)\,dx = 7$, $\int_{4}^{2} g(x)\,dx = 2$, then $\int_{2}^{4} [2 f(x) - 3 g(x) - 5]\,dx$ equals:

a -18 **b** -8 **c** 10 **d** 14

5- The area of the region bounded by the straight lines $y = 2x - 3$, $y = x + 1$, $x = 2$ equals

a 2 **b** 3 **c** $\frac{9}{2}$ **d** 6

6- The volume of the solid generated by revolving the region bounded by the two curves $y = \tan\theta$, and $y = \sec\theta$ and the two straight lines $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$ a complete revolution about x-axis approximated in cubic units squards:

a $\frac{\pi}{6}$ **b** $\frac{\pi}{3}$ **c** $\frac{2\pi}{5}$ **d** 2π

Second: Answer three question only of the following :

2 **a** Find the derivative of y with respect to x where: $y = x^2 \ln x$

b If $f(x) = \sqrt[3]{(x-4)^2}$, find the convexity intervals upwards and downwards and the inflection points (if existed) to the curve of the function f

3 **Find**

a 1- $\int x (x-5)^3 \, dx$ 2- $\int 4x e^{2x} \, dx$

b Find the absolute maximum values of the function f where $f(x) = x^4 - 4x^3$ on the interval [0, 4]

4 **a** The volume of a solid of revolution generated by revolving the region bounded by the curve $y = x^3$ and the two straight lines x = 0 and y = 1 a complete revolution about x-axis is equal to the volume of a cylinder-like wire whose length is 42 units. What is the radius length of that wire?

b The two equal legs of the isosceles triangle with a constant base whose length is L cm/min decrease at a rate of 3 cm/min . what is the rate of the decrease in the area when the triangle becomes an equilateral triangle?

5 **a** Find the area of the region bounded by the two curves $x - y = 0$, $y = 4x - x^2$

b Sketch the curve of the continuous function f which has the following properties:

1- $f(0) = 3$ 2- $f'(2) = f'(-2) = 0$

3- $f'(x) > 0$ when $-2 < x < 2$

4- $f''(x) < 0$ when $x > 0$, $f''(x) > 0$ when $x < 0$.

Test 4

First: Answer the following question:

1 **Choose the correct answer**

1- If $y = \frac{3x-5}{x-2}$, then x = 1 , $\frac{d^3 y}{dx^3}$ equals:

a -12 b -6 c 6 d 12

2- $\int \sec^3 x \tan x \, dx$ equals:

a $\frac{1}{4} \sec^4 x + c$ b $\frac{1}{3} \sec^3 x + c$

c $\frac{1}{2} \tan^2 x + c$ d $-\frac{1}{2} \tan^2 x + c$

3- The normal to circle $x^2 + y^2 = 12$ at any point in it passes thought point

a (2 , 3) b (1, 1) c (0, 0) d (-2, -2)

4- The curve of the function f where $f(x) = (x - 2)e^{x}$ is convex downwards on the interval:

a) $]-\infty, \infty[$ b) $]-1, 2[$ c) $]0, 2[$ d) $]-, \infty[$

5- $\int_{-1}^{3} 3x|x - 4| \, dx$ equals

a) -27 b) -20 c) 20 d) 27

6- When the region bounded by the curve $x = \frac{1}{\sqrt{y}}$, $1 \leqslant y \leqslant 4$ and y-axis revolves a complete revolution about y-axis, then the volume of the solid generated approximated in cabin units equals:

a) $\frac{2}{3}\pi$ b) $3\sqrt{2}\,\pi$ c) $2\pi \log_e 2$ d) $\frac{2}{3}\pi \, \text{Log}3$

Answer three questions only of the following:

2 a) Find: $\int (3x^2 - 4e^2x) \, dx$, $\int \frac{x - 1}{\sqrt{x + 3}} \, dx$

b) If $\sin y + \cos 2x = 0$, prove that: $\frac{d^2y}{dx^2} - (\frac{dy}{dx})^2 \tan y = 4 \cos 2x \sec y$

3 a) If $\int_{1}^{4} f(x) \, dx = 7$, $\int_{4}^{1} g(x) \, dx = 3$, calculate the value of

$\int_{1}^{4} [f(x) + 2g(x) - 4] \, dx$

b) If the curve of the function f where $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum value at (2, 4) and an inflection point at (1, 2), find the equation of the curve.

4 a) Find the area of the region bounded by the curve $\sqrt{x} + \sqrt{y} = 1$ and the two straight lines $x = 0$, $y = 0$

b) Graph the curve of the continuous function f which satisfies the following properties:

$f(4) = 2f(3) = 4$, $f(2) = 0$

$f'(x) < 0$ when $x > 4$ or $x < 2$,

$f''(x) < 0$ when $x > 3$, $f''(x) > 0$ when $x < 3$

5 a) Prove that the volume of the solid generated by revolving the region bounded by the two curves $y = \frac{4}{x}$ and $y = 5 - x$ just one revolution about x-axis equals 9π of the cubic units

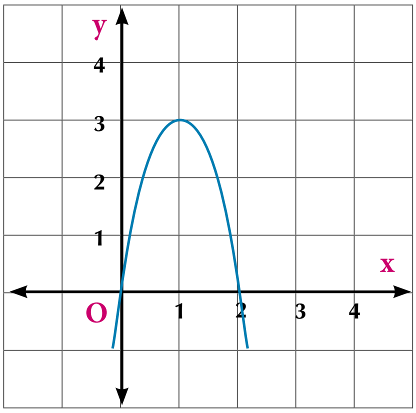
b) If A is the area of the part bounded by two concentric circles whose radii lengths are r_1 and r_2 where $r_2 > r_1$, find the rate of change of A with respect to time at any moment at which $r_2 = 10$ cm, $r_1 = 6$cm, if known that at this moment r_1 increases at a rate of 0.3cm/s and r_2 decreases at a rate of 0.2 cm/s.

Test 5

First: Answer the following question:

1 The opposite figure shows the curve f' (x) of the function f where f (x) = a x^3 + b x^2 , a , b are two constants

Complete:



a The function f is decreasing for each x ∈ ....

b The curve of f has critical points when x ∈ ....

c The curve of f is convex upwards on the interval ....

d There is a local minimum value of the function f when x = ...

e f (1) = ...

f The area of the region bounded by the curve of the function f and the two straight lines x = 2 and y = 0 in square units equals: ....

Second: Answer three questions only of the following:

2 **a** Find:

1 - $\int \csc^2 \frac{x+5}{2}\, d\,x$ **2 -** $\int \frac{5x}{3x^2-1}\, d\,x$

b The function f where f(x) = x^3 - 6 x^2 + 9 x -1

1- Determine the increasing and decreasing intervals of function f

2- Find the absolute maximum values of the function f in the interval [0 , 2]

3 **a** If f (x) = 4 + cot x - $\sec^2$ x, find the equation of the normal to the curve of the function f at a point lying on the curve and its x-coordinate equals $\frac{\pi}{4}$

b An empty tank whose capacity is 10 cubic meters. if the water is poured gradually in that tank at a rate of (2 n + 3) cubi cm/m where t time in minutes, find the time needed to fill the tank.

4 **a** **Find:** $\lim_{x \to \infty} \left(\frac{2x-1}{2x+1}\right)^{2x}$

b A rectangle - like poster contains 800 cm^2 of the printed material where the widths of both lower and upper margins are 10 cm and the two side margins are 5cm. what are the two dimensions of the posters which make its area as minimum aspossible

5 **a** Find the volume of the solid generated by revolving the region bounded by the curve y = 4 - x^2 and the two positive parts of the axes of coordinates a complete revolution about x-axis.

b If f (x) = x^3 + a x^2 + b x + 4 where a and b are two constants,
Find the two values of a and b if the function f has a local minimum value when x = 2 and an inflection point when x = 1, then sketch the curve of the function f

Test 6

First: Answer the following question:

1 **In each of the following phrases, choose A if the phrase is true and B if the phrase is false.**

1 - The local maximum value of the function is greater then its local minimum value (A) (B)

2 - The rate of change of $\sqrt{n^2+3}$ with respect to $\frac{n}{n+1}$ is: $\frac{n(n+1)^2}{\sqrt{n^2+3}}$ (A) (B)

3 - If $\sqrt{y} - \sqrt{x} = 2$, then: $\frac{d^2 y}{d x^2} = \frac{-1}{x\sqrt{x}}$ (A) (B)

4 - $\int \frac{x-4}{(x-2)^6} dx = \frac{7(x-4)^2}{2(x-2)^7} + c$ (A) (B)

5 - If $y = x \log_e x - x$, then: $\frac{dy}{dx} = \log_e x$ (A) (B)

6 - If (a, f(a)) is an inflection point to the curve of the continuous function f, then: f''(a) = zero (A) (B)

Second: Answer three questions only of the following:

2 **Find :**

a $\int \frac{7x^3}{2-5x^4} dx$, $\int (3e^{-5x} + \frac{\pi}{x}) dx$

b If $y = a e^{x^2+1}$ prove that : $\frac{d^3 y}{d x^3} = 4xy(3+2x^2)$

3 **a** Find: $\int \cot x \csc^3 x \, dx$

b If I is the distance between point (1, 0) and point (x , y) lying on the curve $y = \sqrt{x}$, find the coordinates of point (x , y) at which I is as minimum as possible .

4 **a** Identify the absolute extrema values of the function f where f(x) = |x| (x-4) in the intervals [-1, 3]

b If the slope of the tangent to the curve y = f(x) at any point on it equals 6x2 + b x and f(0) = 5 , f(2) = -3 , find the value of the constant b, then sketch the curve of the function f.

5 **a** Find the rate of change of $\log_e(9+x^3)$ with respect to x^2+3 and x = 1

b If a (0, 3) , b (1 , 4) , c (2, 0) , use the integration to find :

First: Surface area of △ A B C .

Second: the volume of the solid generated by revolving △A B C a complete revolution about y-axis .

Test 7

First: Answer the following question :

(1) In each of the following phrases, choose A if the phrase is true and B if the phrase is false.

1 - If $y^2 = 3x^2 - 7$, then: $\frac{dy}{dx} = \frac{y}{3x}$ (a) (b)

2 - The function f : $f(x) = x^3 - 3x + 1$ has an inflection point which is : (0, 1) (a) (b)

3 - $\frac{d}{dx}$ [cot (cos 3x)] = 3 sin 3 x $\csc^2$ (cos 3 x) (a) (b)

4 - $\int (1 - \cos x)^4 \sin x \, dx = -\frac{1}{5} (1 + \cos x)^5 + c$ (a) (b)

5 - $\lim_{x \to \infty} (1 + \frac{5}{x})^x = e^5$ (a) (b)

6 - $\int (\frac{2e}{x} + \frac{x}{e}) \, dx = 2e \log_e |x| - \frac{x^2}{e} + c$ (a) (b)

Second: Answer three questions only of the following:

(2) Find :

a $\int x \sin x \, dx$ and $\int_{-1}^{1} \sqrt{x^4 + x^2} \, dx$

b Find the equation of the tangent to the curve $y = \log_e (2 - \sqrt{2} \cos x)$ at the point lying on it and its x-coordinate equals $\frac{\pi}{4}$.

(3) a Identify the convexity intravels downwards and the inflection points (if existed) to the curve of the function f where $f(x) = (x - 1)^4 + 3$

b A cuboid of metal whose base is square like. If the side length of the base increases at a rate of 0.4/c and the height decreases at a rate of 0.5 cm /c , find the rate of change of the volume when the side length of the base is 6cm and the height is 5cm.

(4) a If $f(x) = \int_0^3 x \sqrt{x + 1} \, dx$

b A rectangle - like playground in which two opposite sides end in a semi circle outside the rectangle of a diameter length equal to the length of this side. If the perimeter of the playground is 400 meters, prove that the surface area of the playground is as maximum as possible when the ground is a circle - like , then find its radius length.

(5) a If $f(x) = x^3 - 3x + 3$, find :

First: the absolute extrema value of the function f in the interval f [0, 2]

Second: the area of the region bounded by the curve of the function f and the straight lines x = 0 , x = 2 , y = 0

b Find the volume of the solid generated by revolving the region bounded by the curve x y = 2 and the two straight lines x = 1 and x =2

Test 8

First: Answer the following question :

1 Complete the following:

a If $x^3 y^2 = 1$, then : $[\frac{dy}{dx}]_{y=1} =$ ....

b $\frac{d}{dx} [7 e^{\sec x}] =$ ....

c The function $f : f(x) = x^3 - 3x - 1$ has an inflection point which is : ....

d If f is a continuous function on the interval [2 , 7], then : $\int_2^7 f(x)\, dx + \int_7^4 f(x)\, dx$

= ....

e The area of the region bounded by the two curves $y = x^2$ and $y = 4x$ equals ....

.... squared units

f If $y = x^2 \log_e \frac{x}{a}$, $a \neq 0$, then: $[\frac{d^3 y}{d x^3}]_{x=4} =$ ....

Second: Answer three questions only of the following:

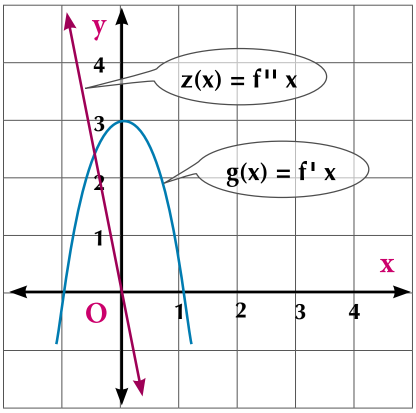
2 Find:

a Find : $\int \frac{(x+3)^3 - 27}{x}\, dx$, $\int x^2 e^{-x}\, dx$

b Find the equation of the tangent to the curve of the function f where $f(x) = 2 \tan^3 x$ at the point lying on the curve of the function f and its x - coordinate equals $\frac{\pi}{4}$

3 **a** Find $\int_0^5 |x - 2|\, dx$

b The opposite figure shows the two curves of the two functions g and h where :
$g(x) = f'(x)$, $z(x) = f''(x)$ and
f is a polynomial function at the variable x.
Sketch the curve of f know that it passes through the two points (-1, 0) , (1, 4)



4 **a** Identify the absolute extrema values of the function f in the interval [0, 2] where $f(x) = 3\sqrt{4 - x^2}$

b A 5-meter rod is fixed by a hinge to the ground at its base. If its top rises up by a winch at a rate of 1m/m, find the rate of decreasing the projector length of the rod on the ground when the height of the top is 3 meters.

5 **a** If a trapezoid is drawn in a semi-circle such that is base is the diameter of the semi-circle, determine the measure of the angle of the trapezoid base such that its area is as maximum as possible.

b If a is the region bounded by the curve $x\,y = 4 + x^2$ and the straight lines $x = 1$, $x = 4$ and $y = 0$, find :

First: Area of region a in square units to the nearest unit .

Second: the volume of the solid generated by revolving the region a about x-axis .

Test 9

First: Answer the following question :

1 Choose the correct answer

1- If $x = 2n^2 + 7$, $y = \sqrt{n^3}$, n = 1, then: $\frac{dy}{dx}$ equals :

a $\frac{3}{8}$ **b** $\frac{3}{4}$ **c** 2 **d** 6

2- The curve of the function f is convex downwards on $\mathbb{R}$ if f(x) equals :

a $2 - x^2$ **b** $2 + x^3$ **c** $2 - x^4$ **d** $2 + x^4$

3- If the cure of the function f : $f(x) = x^3 + k\,x^2 + 4$, $k \in \mathbb{R}$ has an inflection point when x = 2 , then k equals:

a -6 **b** -3 **c** 6 **d** 9

4- If f is a continuous function on $\mathbb{R}$, $\int_{-1}^{3} f(x)\,dx = 7$, $\int_{5}^{3} f(x)\,dx = -11$, then : $\int_{-1}^{5} f(x)\,dx$ equals:

a - 4 **b** 18 **c** -18 **d** 77

5- $\int_{-1}^{3} |x - 1|\,dx$ equals:

a -6 **b** 0 **c** 4 **d** 8

6- The area of the region bounded by the curve $y = x^3$ and the two strought lines y = 0 and x = 2 equals :

a 1 **b** 2 **c** 4 **d** 8

Second: Answer three questions only of the following:

2 **a** Find: $\int \frac{3x}{x^2-1}\,dx$, $\int 9x^2 e^{3x}\,dx$

b Find the measure of the positive angle which the tangent of the curve $y^3 = x^2$ makes with the positive direction of x-axis when x = 8 to the nearest minute .

3 **a** If $\sin x = x\,y$, prove that : $x^2 (y + y'') + 2 \cos x = 2y$

b If the curve $y = 2x^3 + 3x^2 + 4x + 5$ has two parallel tangent; one of them touches the curve at point (-1, 2), find the equation of the other tangent.

4 **a** A balloon rises up vertically at a constant rate of 28m/m. If the balloon is observed by a ground observer distant 200m away from the site of launching the balloon, find the rate of change of the angle of elevation of the observer when the balloon is 200m up.

b If the slope of the tangent to the curve of the function f at any point (x,y) on the curve is $3(x^2 - 1)$, find the local maximum and minimum values to the curve of the function f and the inflection points if existed known that the curve passes thought point (-2, -1) , then sketch this curve.

5 The straight line $\overleftrightarrow{AB}$ intersects the curve of the function f at point c (x , y) where x > 0, A (0, 2), B (6, 4) and $f(x) = \frac{9}{x}$, find :

a The equation of the straight line $\overleftrightarrow{AB}$

b The coordinates of point C

c The equation of the normal on the curve of f at point C and prove that it passes through the origin point O

d The volume of the solid generated by revolving the region bounded by the normal $\overleftrightarrow{OC}$ and the curve of the function and the straight line x = 6 a complete revolution about x-axis.

Test 10

First: Answer the following question :

1 **Complete the following :**

a $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x+3} =$ ....

b $\frac{d}{dx}(5 - 2\cot x)^3 =$ ....

c If the function f : $f(x) = k x^3 + 9x^2$ has an inflection point when x = -1, then :k = ...

d $\int_{-1}^{3} (4x^3 - 6x^2 + 5)\, dx =$ ....

e if f is a continuous function on the interval [1 , 4], then $\int_{1}^{4} f(x)\,dx + \int_{4}^{1} f(x)\,dx =$ ....
....

f The area of the region bounded by the two curves $y = x^4 + 1$ and $y = 2x^2$ equals .... square units.

Second: Answer three questions only of the following

2 **a** Find: $\int \tan(3x + 1)\, dx$ and $\int (1 - x^2)(3x - x^3)^5\, ds$

b If the two parametric equations of the function f where y = f(x) are:

$x = 2n^3 + 3$ and $y = n^4$, find each of the following when n = 1 :

First: The equation of the tangent to the curve of the function f **Second:** $\frac{d^2y}{dx^2}$

3 a Investigate the convexity of the curve of the function f where $f(x) = |x^3 - 1|$ and show the inflection points if existed.

b If $\int_{-2}^{3} f(x)\, dx = 9$, $\int_{5}^{3} f(x)\, dx = 4$, find the value of

$\int_{-2}^{5} [\, 3 f(x) - 6x \,]\, dx$

4 a Find the area of the plane region bounded by the two curves $y + x^2 = 6$, $y + 2x - 3 = 0$

b A right circular cylinder- like container of interior height 9cm and the interior radius length of its base is 6cm. A16 - cm metal rod is placed in the container. If the rate of sliding the rod away from the edge of the cylinder is 2cm/s, find the rate of sliding the rod on the cylinder base when the rod reaches the end of its base.

5 a If the rate of change of the slope of the tangent to a curve at any point (x, y) on it is 6 (1 - 2x) on it is x = 1 and the curve has a critical point when x = 1 and the function has a local minimum value equals 4 .

First: find the equation of the normal to the curve when x = -1

Second: sketch the curve of the function and show the maximum and minimum values and the inflection points if existed.

b Find the volume of the solid generated by revolving the plane region bounded by the curves : $y = x^3 + 1$, $y = 0$ and $x = 0$, $x = 1$ a complete revolution about x-axis.

Answers of some exercises

Unit 1

Exercise (1- 1)

1. -1
2. 18
3. $\frac{3}{8}$
4. 10
5. 6
6. $\frac{-10}{27}$
7. $3x^2 - 2 \sec x \tan x$
8. $3 \csc (2 - 3x) \cot (2 - 3x)$
9. $\frac{-1}{x^2} \csc^2 (\pi - \frac{1}{x})$
10. $- \csc^2 x \sec^2 (\cot x)$
11. $-2 \csc^2 x (1+\cot x)$
12. $2 \csc (\pi-2x) \cot (\pi-2x)$
13. $-2 \sin 2x + 15 \csc^2 3x$
14. $3 \sec^2 3x + 2 \csc 2x \cot 2x$
15. $3 \sin 6x + 2 \sec^2 x \tan x$
16. $2 \sec x \sec^2 2x + \tan 2x \sec x \tan x$
17. $\frac{-3}{2\sqrt{x}} \cot^2\sqrt{x} \csc^2\sqrt{x}$
18. $-4 x \csc^2 (1 + x^2) \cot (1 + x^2)$
19. $12 \sec^2 (2x + \pi) \tan (2x + \pi)$
20. $- \frac{\csc x \cot x}{2\sqrt{1 + \csc x}}$
21. $2x \cot 3x - 3x^2 \csc^2 3x$
22. $\frac{\csc x}{\csc x + \cot x}$
23. $- \frac{3(2 x + 3) \csc^2 3 x \times 2 \cot 3x}{(2 x + 3)^2}$
24. $- \frac{2 \sec x \tan x}{(1 + \sec x)^2}$
25. $\frac{-\pi}{4}$

27. a) -2 b) 2

Exercise (1-2)

1. c
2. c
3. d
4. a
5. b
6. $\frac{dy}{dx} = \frac{x}{4y}$
7. $\frac{dy}{dx} = -\frac{x^3}{3y^3} = -\frac{1}{3} (\frac{x}{y})^3$
8. $\frac{dy}{dx} = 1$
9. $\frac{dy}{dx} = - \frac{3(x^2 + 2y)}{2 (3x -2)}$
10. $\frac{dy}{dx} = \frac{y-2x}{2y-x} = \frac{2x-y}{x-2y}$
11. $\frac{dy}{dx} = \frac{-y}{x+\cos y}$
12. $\frac{dy}{dx} = \frac{y \sin x - \sin y}{x \cos y + \cos x}$
13. $\frac{dy}{dx} = \frac{y \csc^2 x + \csc y}{\cot x + x \csc y \cot y}$
14. $\frac{dy}{dx} = \frac{y^2 \cos x - 2 x \sin y}{x^2 \cos y - 2y \sin x}$
15. $\frac{dy}{dx} = \cot 2x \cot 2y$
16. $\frac{dy}{dx} = -\frac{127}{8}$
17. $\frac{dy}{dx} = -\tan\frac{\pi}{3} = -\sqrt{3}$
18. $\frac{dy}{dx} = \frac{1}{2}$
19. (-3)
20. $(-\frac{2}{3})$
21. a) The curve has a vertical tangent when $n = \frac{2}{3}$ or $n = 1$
    b) The curve has a horizontal tangent when $n = -\frac{1}{4}$

Exercise (1-3)

1. $a = 1$, $b = 11$, $c = 19$, $d = -23$
   $y = 67$, $y' = 44$
2. $y''' = 60x^2 - 24$
3. $y''' = \frac{12}{(x + 1)^4}$
4. $y''' = -8\cos (2x - 7)$
5. $y''' = -27 \sin (\pi - 3x)$
6. $y''' = -4\cos 2x$
7. $y''' = \frac{3}{\sqrt{(2x - 5)^5}}$

14. $\frac{2}{27}$
15. $\frac{1}{48}$
16. $-\frac{1}{9}$

Exercise (1 - 4)

1. a) Equation of tangent: $x + y - 10 = 0$
   , Equation of normal: $x - y + 4 = 0$
   b) $12 x - y - 86 = 0$
   $x + 12 y + 17 = 0$
   c) $12 x - y - 38 = 0$
   $x + 12 y + 21 = 0$
2. a) $y - 2 = 4 (x - \frac{\pi}{4})$
   $y - 2 = -\frac{1}{4} (x - \frac{\pi}{4})$
   b) $y + 1 = -3 \sqrt{3} (x - \frac{\pi}{3})$
   $y + 1 = \frac{1}{3\sqrt{3}} (x - \frac{\pi}{3})$
3. a) $2 x - 3y - 26 = 0$ $3x + 2 y = 0$
   b) $x + y + 2 = 0$ $x - y = 0$
   c) $x - 2 y + 5 = 0$ $2 x + y = 0$
   d) $y = 0 , x = \frac{\pi}{2}$

4 a $2x - 3y - 4 = 0$ $3x + 2y - 19 = 0$

b $2x - y - \sqrt{3} = 0$ $\sqrt{3}(x + 2y) = 4$

5 $k = 4$ $y + 2 = 0$

6 8.5 square units

7 at point a (0, 1): $x - y + 1 = 0$

, $x + y - 1 = 0$

at point b (0, -1): $x + y + 1 = 0$

, $x - y - 1 = 0$

Exercises (1 - 5)

1 d 2 b 3 a 4 b

5 $\frac{dy}{dt} = -0.8$ unit/sec

6 $\frac{dA}{dt} = 160\pi$ cm^2/sec

7 $\frac{dA}{dt} = -3\sqrt{3}$ cm^2/ sec

8 $\frac{dy}{dt} = 50$ c gm/cm^2/sec

9 $\therefore \frac{dr}{dt} = \frac{-1}{20\pi}$ cm /sec $\therefore \frac{dA}{dt} = -4$ cm^2 /sec

10 -3cm/ min $\therefore \frac{d\theta}{dt} = -0.01$d / f

11 48 m/ min 13 -60 cm^2 / h

Answers of general exercises

1 d 2 c 3 c 4 b

5 b

6 a $\frac{dy}{dx} = 1 - 2\csc^2 2x$

b $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}} + 5\sec 5x \tan 5x$

c $\frac{dy}{dx} = 2 + 10\pi^2 x \sin(\pi x)^2$

d $\frac{dy}{dx} = -2\pi \sin 2(\pi x + 1)$

e $\frac{dy}{dx} = 0$

f $\frac{dy}{dx} = 2\sec 2x(2\sec^2 2x - 1)$

7 $\frac{dx}{d\theta} = -2\sqrt{3}\, r$

8 a $\frac{dy}{dx} = \frac{x}{3y}$ b $\frac{dy}{dx} = \frac{-5x}{18y^2}$

c $\frac{dy}{dx} = \frac{x - y}{x - 2y}$ d $\frac{dy}{dx} = \frac{3 - x}{y + 2}$

e $\frac{dy}{dx} = -\frac{y + \cos x}{x}$

f $\frac{dy}{dx} = \cot x \cot y$

9 a $\frac{dy}{dx} = \frac{1}{5}(2x + 1)(x + 2)^2$

b $\therefore \frac{dy}{dx} = \frac{-6}{25}$

11 a $x\sqrt{3} - 9y + 24 = 0$

$3\sqrt{3}x + y - 12 = 0$

b $2x - y = 0$ $4x + 8y - 5\pi = 0$

12 a $3x + y - 10 = 0$ $x - 3y = 0$

b $x + 2y + 1 = 0$ $2x - y - 3 = 0$

13 1 square units

16 99cm/sec 18 -1 unit /sec

19 -8 cm^2 /sec $\frac{8}{3}$ sec $\frac{784}{3}$ cm^2

20 $\frac{3}{4}$ k unit /sec 21 10cm

22 $-\frac{5}{3}$ m/ min $\therefore \frac{d\theta}{dn} = -\frac{5}{3}$rad / min

23 -12 cm^3/sec 24 $-\frac{1}{72\pi}$ m / min

25 0.14 rad / min 26 -1rad / sec

27 $\frac{3}{14}$ square units / sec

Answer of accumulative test

1 c 2 a 3 c 4 d

5 $\frac{6n^2 - 6n + 2}{(1 - 2n)^3}$

6 $x - 3y + 48 = 0$ 8 20

9 a $x - 6y + 18 - \frac{\pi}{4} = 0$

b 19.3 cm^2 /sec

10 $-\frac{20\sqrt{3}}{3}$ cm/ sec

Unit 2

Exercises 2 - 1

1 d 2 c 3 b 4 b

5 e 6 e 7 e^x 8 e^4

9 2 10 1 11 e^4 12 -1

13 2 ln a 14 e^3 15 e^{-1} 16 $\frac{3}{2}$

Exercises (2 - 2)

(1) b (2) d (3) d (4) c

(5) $15x^4 e^{3x^5}$ (6) $(2x - 1)e^{x^2 - x}$

(7) $-2(3^{x-1})^{-2} \ln 3$ (8) $5x\, e^{\frac{1}{2}(5x^2 - 3)}$

(9) $\frac{2}{2x - 7}$ (10) $\frac{2(x+1)}{x(x+2)}$

(11) $\frac{x + 14}{x(x+7)}$ (12) $x[1 + 2\ln x]$

(13) $\frac{8}{4x+9} \log_3 e$

(14) $e^{3x}\left[\frac{3}{\log x} - \frac{\log e}{x_{-}(\log x)^2}\right]$

(15) $e^x \sec e^x \tan e^x$

(16) $6e^{3x} - \frac{5}{x} - \log_2 e$ (17) -1.57

(18) $\frac{5}{2}$ (19) -2.64

(20) $\frac{dy}{dx} = -y$ (21) $\frac{dy}{dx} = \frac{-58y}{x^2}$

(22) $x^{\sin x}\left[\frac{\sin x}{x} + \cos x \ln x\right]$

(23) $e^x \times e^{e^x}$ (24) $e\, x^{e-1} \times e^{xe}$

(25) $x^{\frac{1}{x}-2}(1 - \log_e x)$

(26) $\frac{3n^2}{2e^{2n}}$ $\frac{3}{2} \times \frac{n(1-n)}{e^{4n}}$

(27) $\frac{1}{3} \times n^2$ $\frac{1}{9} \times n^2$

(28) $\frac{1}{2}$ (31) $x = \frac{2}{3}$

(32) $e^x + 3y - e - \frac{9}{e} = 0$ (33) $\frac{120}{e^3}$

(34) 12.8 gm/ day 8.7 gm/ day
8 gm / day

Exercises 2-3

(1) d (2) c (3) a (4) c

(5) $\frac{1}{4} e^{4x} + c$ (6) $x^3 + 2e^x + c$

(7) $4 \ln |x| + e^{-x} + c$

(8) $-\frac{1}{3} e^{1-3x} + c$ (9) $\frac{7}{9} e^{3x-4} + c$

(10) $\frac{2}{3}(e^x + 1)^3 + c$

(11) $\frac{1}{2} e^{2x} + 2e^x - 4e^{-x} + c$

(12) $\frac{1}{3} e^{x^3+1} + c$ (13) $2\ln(e^x + 1) + c$

(14) $\frac{1}{4} \ln |4x - 1| + c$ (15) $\frac{1}{2} \ln(x^2 + 1) + c$

(16) $\ln |\tan x| + c$

(17) $\ln |\sin x - \cos x| + c$

(18) $\ln |1 + \sin x| + c$ (19) $\ln |\sec x - 1| + c$

(20) $\ln |\ln x| + c$

(21) $2\ln|x + 1| + \frac{2}{x+1} + c$

(22) $\frac{1}{3}(\ln x)^3 + c$ (23) $\ln |x^3 - 1| + c$

(24) $\ln |x^3 - 5x + 1| + c$

(25) $4\ln|x| + e^x + c$

(26) $4 \ln |\ln 3x| + c$ (27) $\frac{1}{3}(1 + \ln x)^3 + c$

(28) $f(3) \simeq 4.11$

Answers of general exercises

(1) d (2) a (3) c (4) d

(5) s.s = {103} (6) s.s = {25}

(7) s.s = {0.399} (8) s.s = {1.91}

(9) 2.197} = م.ح} (10) s.s = {1.654}

(11) e^6 (12) e^{-1} (13) e^{-2} (14) $7e^x$

(15) $6x\, e^{3x^2+1}$ (16) $2x\, e^{-3}$

(17) $\frac{dy}{dx} = \frac{2x}{x^2+3}$ (18) $\frac{dy}{dx} = 2 + \ln x^2$

(19) $e^x\left[\frac{2x}{x^2+1} + \ln(x^2+1)\right]$

(20) $\frac{dy}{dx} = \frac{e^x}{(e^x+1)^2}$

(21) $= 3x^2$ (22) 0

(23) $\frac{d^2y}{dx^2} = \frac{3}{4e^n}$ (24) $\frac{d^2y}{dx^2} = \frac{-4}{3n^6}$

(25) $\frac{d^3y}{dx^3} = \frac{2}{x^3}$ (26) $\frac{d^3y}{dx^3} = 6 + 8e^{2x}$

(27) $\frac{d^3y}{dx^3} = -\frac{1}{x^2 \ln 10}$

(30) $\frac{dy}{dx} = 2^x[1 + (x+1)\ln 2]$

(31) $\frac{dy}{dx} = 3^{x^2+2}\left[\frac{x^2+3}{x} + 2x \ln x\right]$

32 $-[\sin x \ln(1-3x) + \frac{3\cos x}{1-3x}](1-3x)^{\cos x}$

33 $\frac{dy}{dx} = e^x(\frac{1}{x} + \ln x)x^{e^x}$

34 $x = \frac{1}{\sqrt[3]{e}}$

35 $x = \sqrt{e}$

36 $x = 3$

37 $x = \sqrt{2}$

38 $\frac{dy}{dx} = \frac{e-1}{e}$

39 $3\ln|x^2 + 3x| + c$

40 $2\ln|x| + c$

41 $3\ln|x| - 2e^{-2x} + c$

42 $e^3\ln|x| + \frac{1}{2}x^2\ln^3 + c$

43 $\frac{1}{4}x^4 - \frac{2}{3}e^x + c$

44 $\frac{1}{2}\cot^2 x - \ln|\sin x| + c$

45 $\simeq 7.456$ length unit

46 $3x - y - 1.52 = 0$ $\quad x + 3y - 15.44 = 0$

47 $y = 8\ln|x| - 8\ln 4 + 2$

48 $x \in \{1.65, -1.65\}$

Answers of accumulative test

1 c
2 d
3 b
4 c

5
a) $y' = 6x^2(x^3+1)$

b) $y' = \frac{2}{x} - \frac{1}{4}e^{\frac{1}{4}x}$

c) $y = 2(x - \frac{1}{x})$

7
a) $x^2 + 3\ln|x| + c$

b) $-2e^{-\sqrt{x}} + c$

c) $\frac{3}{2}\ln|\ln x| + c$

8 $f(x) = 7(x + 1 - \ln 2) - 2e^x$

Unit 3

Exercises (3-1)

1 f is decreasing on]- ∞, 2[
f is increasing on]2 , ∞[

2 f is decreasing on]- ∞, 3[
f is increasing on]3, ∞[

3 f is increasing on]- ∞, 0[
f is decreasing on] 0 , 4[
f is increasing on] 4 , ∞ [

4 f is decreasing on]- ∞,- $\sqrt{3}$ [
f is increasing on] - $\sqrt{3}$, $\sqrt{3}$ [
f is decreasing on] $\sqrt{3}$, ∞ [

5 f is decreasing on]- ∞, -1[
f is increasing on] -1 , ∞[

6 f is increasing on]- ∞, 2[
f is decreasing on] 2 , ∞[

7 f is increasing on]- ∞, 0[
f is increasing on] 0 , ∞[

8 f is increasing on]- ∞, -2[
f is increasing on] -2 , ∞[

9 f is increasing on]1, 2[,
f is decreasing on] 2 , ∞[

10 f is increasing on] 0 ,∞ [

11 f is increasing on]- ∞, 0[
f is decreasing on] 0 , ∞[

12 f is decreasing on] - ∞, ∞ [

13 f(x) is increasing on] 0 , $\frac{\pi}{4}$[

14 f is decreasing on] 0 , $\frac{\pi}{2}$[
f is increasing on]$\frac{\pi}{2}$,$\frac{3}{2}\pi$[
f is decreasing on]$\frac{3}{2}\pi$, 2π[

Exercises 3-2

1 $f(0) = 0$ local maximum value
$f(2) = -4$ local minimum value

2 $f(-1) = -2$ local maximum value
$f(1) = 2$ local minimum value

3 $f(0) = 0$ local maximum value
$f(1) = -1$ local minimum value

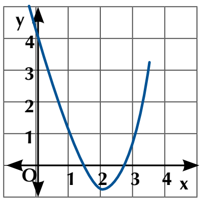
4 $f(-2) = 6$ local maximum ,
$f(0) = 2$ local minimum

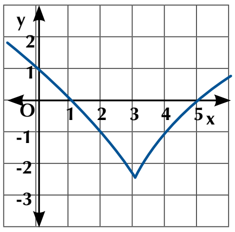
5 $f(-1) = -1$ local minimum,
$f(0) = 0$ local maximum
$f(1) = -1$ local minimum

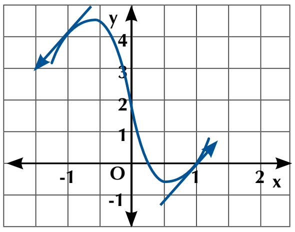
6 $f(\frac{-2}{\sqrt{3}}) = \frac{-16}{3\sqrt{3}}$ local minimum
$f(\frac{2}{\sqrt{3}}) = \frac{16}{3\sqrt{3}}$ local maximum

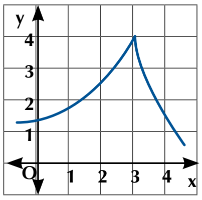
7. $f(-1) = 2$ local maximum,
   $f(1) = -2$ local minimum
8. $f(0) = 3$ local maximum
9. $f(-2) = 0$ local minimum value
10. $f(2) = 3$ local minimum value
11. $f(-1) = -3$ local maximum value
    $f(3) = 5$ local minimum value
12. No local maximum or local minimum
13. $f(0) = 4$ local maximum value
14. $f(2) = e^2$ local maximum value
15. $f(0) = 2$ local minimum value
16. $f(1) = 1$ local minimum value
17. $f(2) = 8 \ln -4 \simeq 1.55$ local maximum value
18. $f(1) = 0$ local minimum value
19. The function has absolute maximum value $=3$, the value of absolute minimum $= -1$
20. The function has absolute maximum value $= 2$, the value of absolute minimum $= 1$
21. The function has absolute maximum value $= \sqrt{2}$, the value of absolute minimum $= -\sqrt{2}$
22. The function has absolute maximum value $\simeq 0.37$, the value of absolute minimum $= 0$
23. $a = 1$, $b = -9$, $c = 15$, $d = 0$

Exercises (3-3)

1. **a** $[-4, 5]$ **b** $\{-2,2\}$ **c** $]0 , 5[$
   d $]-4, 0[$ **e** $(0 , 0)$ **f** 2 **g** 8
2. The curve is convex upwards for each $x \in R$, no inflection points
3. Convexity : upwards on $]-\infty, 1[$
   downwards on $]1 , \infty[$, $(1,-1)$ inflection point
4. Convexity: upwards on $]2 , \infty[$, downwards on $]-\infty, 2[$, $(2, 46)$ inflection point.
5. Convexity : upwards on $]\frac{-2}{\sqrt{3}} , \frac{2}{\sqrt{3}}[$
   downwards on $]-\infty , \frac{2}{\sqrt{3}}[$, $]\frac{-2}{\sqrt{3}}, \infty[$
   inflection points: $(\frac{-2}{\sqrt{3}}, \frac{64}{9})$, $(\frac{2}{\sqrt{3}}, \frac{64}{9})$
6. Convexity : upwards on $]-1, 0[$
   downwards on $]-\infty, -1[$, $]1,\infty[$
   inflection points: $(-1, \frac{3}{2})$, $(1, \frac{3}{2})$
7. Convexity : upwards on $]-2, 2[$
   downwards on $]-\infty, -2[$, $]2, \infty[$
   no inflection points
8. Convexity : downwards on $]-\infty, 4[$
   upwards on $]4 ,\infty[$ no inflection point because there is no tangent to the curve at point $(4,4)$
9. Convexity : upwards on $]-\infty, 0[$, $]0,\infty[$
   no inflection points
10. $\theta = \tan^{-1}(1) = \frac{\pi}{4}$
12. $a = 3$, $b = 4$
13. 

14. 

15. 

16. 

17. (0 , 5) with y-axis
(1, 0) , (5 , 0) with x-axis.
f(3) = -4 local minimum
18. (0 , 3) with y-axis
($\sqrt{3}$, 0) , (-$\sqrt{3}$, 0) with x-axis.
f(0) = 3 local maximum
19. Inflection point: (1,1)
f(0) = 3 local maximum
f(2) = -1 local minimum
additional points : (-1, -1) , (3, 3)
20. Inflection point: (0 , 2)
f(1) = $\frac{4}{3}$ local minimum f(-1) = $\frac{8}{3}$ local maximum, additional points: (3 , 8) , (-3 , -4)
21. Inflection point: (0 , 1)
f(2) = -1 local minimum
f(-2) = 3 local maximum
additional point: (4 , 3) , (-4 , 1)
22. Inflection points: (2 , -2)
f(1) = -4 local minimum
f(3) = 0 local maximum
intersection with axes: the curve passes through origin point
additional point: (4 , -4)
23. Inflection points : (0 , 2)
f(1) = 4 local maximum
f(-1) = 0 local minimum
intersection points with axes:
(2 , 0) , (-1 , 0) with x-axis , (0 , 2) with y-axis
additional point: (-2, 4)
24. Inflection point: (0 , 2)
f(2) = 0 local minimum
f(-4) = 0 local maximum
intersection with axes: (0 , 2) with y-axis
(-4, 0) , (2 , 0) with x-axis.
25. inflection points : (1 , -2) only
f(2) = -4 local minimum
additional point: (-1 , 3)
26. f(0) = 4 local maximum or no inflection points additional point = (5 , 5)

Exercises (3-4)

1. 15 , 15
2. 2 , 3
3. (1)
4. 900 square meters.
5. r = 7.5 cm
6. 20 , 20, 20 cm.
7. 5 $\sqrt{2}$ cm, 5 $\sqrt{2}$ cm
8. 80000 square meters
9. $\frac{h}{r} = 2$
10. $\frac{44100}{\pi}$ square meters
11. 15$\sqrt{2}$, 15$\sqrt{2}$ cm
12. Dimension of the containers 3 , 9, 18 cm
14. (4 , 4) , (4 , -4)
15. $\frac{6}{\sqrt{5}}$ length unit
16. 90°
17. 2 $\sqrt{2}$ Ampere
18. 2250
19. x = $\frac{10}{3}$ cm

Answer of the general exercises

1. c
2. b
3. d
4. a
5. d
6. f is decreasing on] - ∞, 3 [is increasing on] 3 , ∞ [
7. f is increasing on] - ∞, 0 [,] $\frac{1}{2}$, ∞[
is decreasing on] 0 , $\frac{1}{2}$ [
8. f is increasing on] - ∞, ∞[
9. f is increasing on]- ∞, 2[
is decreasing on] 2 , ∞[
10. f is increasing on] - ∞, 0 [,]$\frac{4}{3}$, ∞[
is decreasing on] 0 , $\frac{4}{3}$ [
11. f is decreasing on]- ∞, 3[
is increasing on]3, ∞[
12. f is decreasing on] - ∞, 0 [,]0 ,∞[
13. f(x) = $\frac{x}{x^2}$ f is increasing on]- ∞, -2 [,] -2, ∞[
14. f is increasing on] - ∞, 1[,]1 , ∞[
15. Convex downward on x ∈ R
16. Convex upwards on] - ∞, 1[
downward on] 1 ,∞ [, (1 , -2) inflection point
17. Convex upwards on]- ∞, 2[
downward on]2 , ∞[(2 , -7) inflection point

18. convex upwards on] - ∞, 0 [,] 3 , ∞[
convex downward on]0 , 3[
inflection points: (0 , 0) , (3 , 81)

19. convex downward on] - ∞, 0 [,]2, ∞[
upwards on]0 , 2[, inflection points: (2, 0)

20. convex upwards on]-∞,0[
downward on] 0 , ∞[
inflection points: (0, 7)

21. f(-3) = 13 local maximum value

22. f(0) = 7 local maximum value
f(1) = 6 value of local minimum.

23. $f(\frac{2}{3}) = \frac{32}{27}$ local maximum value
f(2) = 0 value of local minimum

24. f(-2) = -8 value of local minimum
f(0) = 8 local maximum value
f(2) = -8 local minimum value

25. f(0) = -1 value of local minimum.

26. f discontinuous x = 2
f(1) = 2 local maximum value
f(3) = 6 value of local minimum

27. $f(-1) = -\frac{1}{2}$ value of local minimum
$f(9) = \frac{1}{18}$ local maximum value.

28. f (2) = 0 value of local minimum.

29. -1.6 value of local minimum

30. the function has absolute minimum value = -9
absolute maximum value = 18

31. absolute minimum value = 0
absolute maximum value = 81

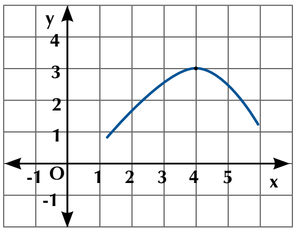
32. absolute maximum value = 2
absolute maximum value = 18

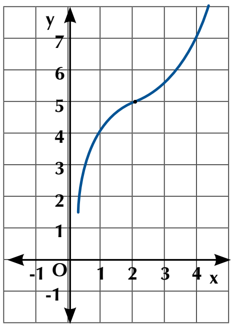
33. absolute maximum value = 2
absolute minimum value = $\frac{7}{8}$

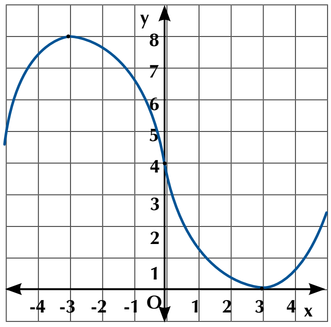
34. absolute maximum value = 16 ,
absolute minimum value = -6

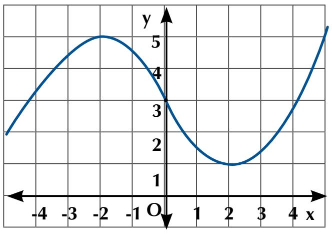
35. 36 absolute maximum value, -4 absolute minimum value

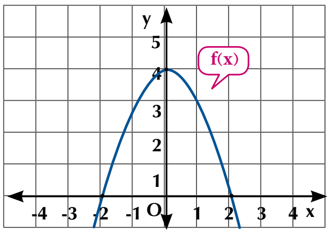
36. -54 absolute minimum value , 3 absolute maximum value

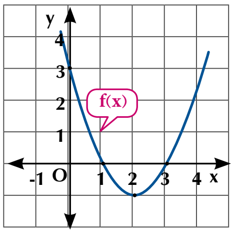
37.


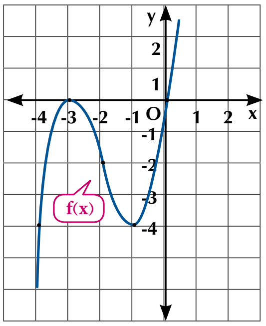
38.


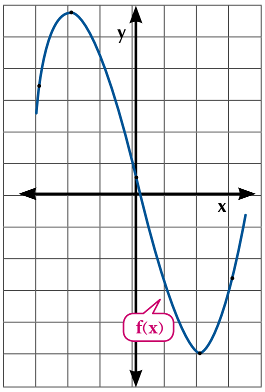
39.


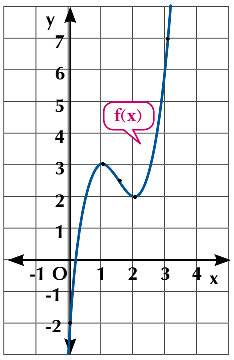
40.


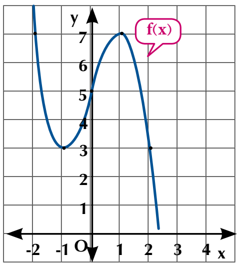
41.


42.


(43)


(44)


(45)


(46)


(47) $\therefore$ d = 6 , c = 0 $\quad \therefore$ a = 2 , b = -3

(48) The two dimensions of the rectangle are 5 meter , 5 meter

(49) (b) 28 square units

(50) $\mathbb{R} = 250\pi\sqrt{3}$ cm^3

(51) $10\sqrt{2}$ cm

(53) 2 square units (54) $\frac{3}{2}\sqrt[3]{2}$ meter

Answer of accumulative test

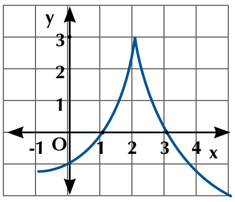
(1) (a)]2 ,∞ [(b) x = 2 (c) x = 5
(d)] -∞ , -1[,] 5, ∞[

(2) (a) When $x = \frac{-b}{a}$ there is local maximum value

(b) f is increasing on $]-\infty , \frac{5}{2}[$
is decreasing on $]\frac{5}{2} , \infty[$

(3) (-1, 0) , (0, 1) , (1, 0) critical point
increasing intervals :]-1, 0[,]1, ∞[
decreasing intervals :] -∞ , -1[,] 0, 1[

(4) critical point : (2, 8)
convex upwards on]-∞, 2[
convex downwards]2, ∞[inflection point : (2, 8)

(5)


(6) (a) a = -3 , b = 3
(b) Convex : upwards on]-1 ,∞[
] 1 , ∞[there is not an inflection point

(7) (2 , 3) (8) $(\frac{7}{4} , 0)$

Unit 4

Exercises 4 -1

(1) b (2) d (3) b

(4) $\frac{1}{30}(x - 2)^5(5x + 2) + c$

(5) $\frac{1}{30}(x - 2)^4[5x^2 + 4x + 2] + c$

(6) $\frac{1}{84}(x^2 - 1)^6(6x^2 + 1) + c$

(7) $\frac{2}{15}(x + 4)^{\frac{3}{2}}(3x - 8) + c$

(8) $\frac{2}{35}(x + 1)^{\frac{5}{2}}(5x - 9) + c$

(9) $\frac{1}{70}(x^2 + 3)^5(5x^4 - 5x^2 + 3) + c$

(10) $\frac{1}{6}\ln(3x^2 + 2) + c$

(11) $x - \ln|x + 1| + c$

12. $\frac{2}{3}(x+5)\sqrt{x-1}+c$
13. $\frac{1}{15}(3x^2+2x+2)\sqrt{2x-1}+c$
14. $-\frac{1}{2}e^{-x^2}+c$
15. $-\ln|e^{-x}+x|+c$
16. $\frac{1}{2}(\ln 5x)^2+c$
17. $\frac{1}{4}(\ln x)^4+c$
18. $\ln|\ln x|$ c
19. $(2x-1)e^{2x}+c$
20. $\frac{1}{2}(x^2-1)e^{x^2}+c$
21. $-\frac{2x+1}{4e^{2x}}+c$
22. $\frac{1}{16}x^4(4\ln x-1)+c$
23. $x(\ln x^2-2)+c$
24. $i=x[(\ln x)^2-2\ln x+2]+c$
25. $-\frac{1}{x}(1+\ln x)+c$
26. $\frac{e^{2x}}{4}[2x^2+2x+1]+c$
27. $\frac{1}{4}x^2[2(\ln x)^2-2\ln x+1]+c$
28. $y=\ln|x-3|+3$
29. $y=\frac{2}{15}(x+1)^{\frac{3}{2}}(3x-2)+1$
30. $y=x^3-3x+2$

f(-1) = 4 local maximum value

Exercises (4 -2)

1. f
2. a
3. b
4. b
5. $\sqrt{3}\sin x+c$
6. $-\frac{1}{2}\cot(2x+3)+c$
7. $-\cos x+c$
8. $-\frac{1}{4}\cos 2x+c$
9. $x+c$
10. $x+\frac{1}{2}\cos 2x+c$
11. $-\frac{1}{4}\cos^4 x+c$
12. $\frac{1}{6}\sin^6 x+c$
13. $\frac{1}{3}\sin(x^3+5)+c$
14. $\frac{1}{6}(3+\sin x)^6+c$
15. $\frac{1}{5}\sec^5 x+c$
16. $\sin x-4\tan x+c$
17. $\frac{1}{2}x+\frac{1}{4}\sin 2x+\sin x+c$
18. $\frac{5}{2}x+\frac{1}{4}\sin 2x+\tan x+c$
19. $\tan x-\frac{1}{2}\sin 2x+c$
20. $2y=9+14x+\cos 2x$
21. $y=x^2+8+\tan\frac{x}{2}$
22. $y=2\cot x+3$

Exercises (4 - 3)

1. c
2. d
3. 20
4. 22
5. 16
6. $4(1-\sqrt{2})$
7. $\frac{14}{3}$
8. $\simeq$ 4.43
9. 3
10. $-\frac{1}{5}(4)^4$
11. $\frac{112}{5}$
12. $\frac{4}{\pi}$
13. $\frac{1}{2}$
14. $16-7e^3$
15. a 8 b 12 c 6
16. a 11 b zero c 6
17. 20

Exercises (4 - 4)

1. $5\frac{1}{4}$ square units
2. 3 square units
3. 2 ln 3 square units
4. 6 square units
5. $\frac{9}{2}$ square units
6. $\frac{16}{3}$ square units
7. c
8. d
9. c
10. b
11. 12 square units
12. 7 square units
13. $\frac{38}{3}$ square units
14. $10\frac{2}{3}$ square units
15. 3 square units
16. 2.25 square units
17. 2.5 square units
18. 9 square units
20. 22 square units

Exercises (4 - 5)

1. d
2. b
3. b
4. c
5. 9π cubic units
6. 9π cubic units
7. $\frac{3}{4}$ cubic units
8. 24π cubic units
9. $\frac{\pi}{3}$ cubic units
10. 32π cubic units
11. 8 cubic units
12. $\frac{96}{5}$ cubic units
13. 36π cubic units
14. **First:** 8π cubic units

    Second: $\frac{512}{15}\pi$ cubic units
15. 9π cubic units
16. 50π cubic units

Answers of general exercises

(1) c (2) d (3) a (4) a

(5) c (6) 6x dx (7) $5\,(2x+3)^{\frac{3}{2}}dx$

(8) $3\,(x+\frac{1}{x})^3\,(1-\frac{1}{x})dx$

(9) $-5\,e^{-5x}\,d\,x$

(10) $(2x+3)\,e^{2x^2+3x}\,d\,x$

(12) $\frac{2x}{x^2+1}\,d\,x$ (13) $\frac{-2}{x}\ln\frac{10}{x}\,d\,x$

(14) $-3\tan(3x-1)\csc^2(3x-1)\,d\,x$

(15) $\int_{-2}^{5}x^2\,dx$ (16) $\int_{-4}^{4}\sqrt{x+6}\,dx$

(17) $\int_{3}^{6}k\,dx$ (18) $\int_{-4}^{5}x^2\,dx$

(19) (a) 15 (b) 6 (c) 9 (d) 15

(20) x^3-2x^2+3x-4 (21) $\frac{227}{12}$

(22) $-\frac{7}{3}$ (23) $\frac{23}{3}$ (24) $\frac{46}{5}$ (25) 8π

(26) π (27) $\frac{1}{20}(x-5)^4\,(4x+5)+c$

(28) $-\frac{1}{(1+\sqrt{x})^2}+c$ (29) $\frac{1}{2}\tan(x^2+2)+c$

(30) $\frac{1}{4}\sec^4x+c$ (31) $\frac{1}{2}e^{x^2+3}+c$

(32) $\frac{1}{2}\ln(e^{2x}+x^2)+c$

(33) $\frac{3}{5}(x-1)\,(2x+3)^{\frac{3}{2}}+c$

(34) $\frac{1}{25}x^5\,[\,5\ln x-1]+c$

(35) $e^{2x}\,[\,2x-1]+c$

(36) $\frac{64}{3}$ square units (37) $\frac{46}{3}$ square units

(38) $\frac{22}{3}$ square units

(39) (a) $\frac{2}{3}\pi$ square units

(b) $2\pi\,[2+\ln 3]$ cubic units

(40) $\frac{4\pi}{3}$ cubic units

Answer of accumulative test

(1) b (2) c (3) b (4) c

(5) d (6) d

(7) (a) $\sqrt{x^2+6x}+c$

(b) $\frac{2}{35}\,(5x^2+12x+24)\,(x-3)^{\frac{3}{2}}+c$

(8) 106

(9) (a) $\frac{1}{3}\sec^3x+c$ (b) $\frac{2}{3}\sqrt{(1+\sin x)^3}+c$

(10) (a) $\frac{1}{3}x^3[\ln x^2-\frac{2}{3}]+c$

(b) $\frac{1}{4}e^{2x^2-1}+c$

(11) (a) zero (b) zero

(12) -13 (13) 4 square units

(14) (a) $\frac{4}{3}$ square units (b) $2.4\,\pi$ cubic units

Answer of tests

Test 1

(1)

1	2	3	4	5	6
c	b	b	b	d	c

(2) (a) $\frac{-1}{4}\cos^4x+c$ (b) $\frac{1}{3}$

(3) (a) $14x+5y-6=0$

(b) 2cm/min , 6 minutes

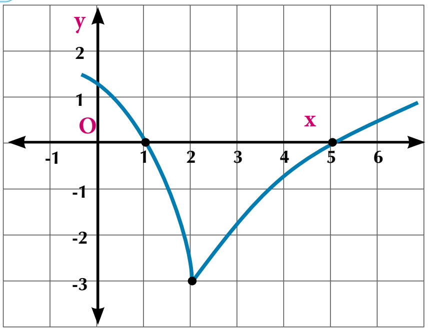
(4) (a) f is decreasing on $]\frac{2\pi}{3},\frac{4\pi}{3}[$

f is increasing on $]0,\frac{2\pi}{3}[\,,\,]\frac{4\pi}{3},2\pi[$

(b) 64 square units

(5) (a) $\frac{27}{5}$ cubic units

(b)



Test 2

(1)

1	2	3	4	5	6
b	c	b	a	c	c

(2) (a) $\frac{1}{80}\,(2x-1)^4\,(8x+1)+c$

(b) $-\frac{1}{4}e^{-2x}\,[\,2x+1]$

3 a $y' = \dfrac{y \sin x - \cos y}{\cos x - x \sin y}$

b absolute minimum value = 5 ,

absolute max. value= 13

4 a f (- 1) = - 1 local minimum

f(1) = 1 local maximum value $-\frac{2}{3}$

b 36 cm^2/min

5 a 9 square units b a = - 6 , b = 9

Test 3

1

1	2	3	4	5	6
b	b	d	c	a	a

2 a $x(1 + \ln x^2)$

b The curve is convex upwards on

] - ∞ , 4[ ]4 , ∞ [no inflection points

3 a $\frac{1}{20}(4x + 5)(x - 5)^4 + c$

$(2x - 1)\, e^{2x} + c$

b Local maximum value = 0 ,

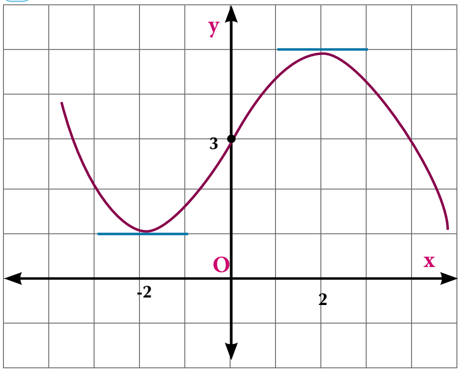
absolute minimum value = - 27

4 a $r = \frac{1}{7}$ length unit

b $L\sqrt{3}$ cm^2 /cm

5 a $\frac{9}{2}$ square units

b



Test 4

1

1	2	3	4	5	6
b	b	c	d	c	c

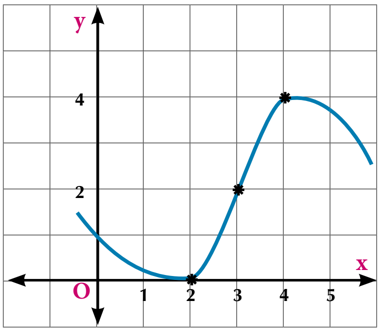
2 a $x^3 - 2e^{2x} + c$

$\frac{2}{3}(x - 3)\sqrt{x + 3} + c$

3 a - 11 b $f(x) = 3x^2 - x^3$

4 a $\frac{1}{6}$ square units

b



5 b 7.6π cm^2/sec

Test 5

1 a x∈] -∞ , 0[, x∈] 2 , ∞[

b x∈ { 0, 2} c] 1 , ∞[

d x = 0 e f(1) = 2

f 4 square units

2 a $-2 \cot \frac{x + 5}{2} + c$ $\frac{5}{6} \ln |3x^2 - 1| + c$

b f is decreasing on] 1 , 3[,

is increasing on] - ∞, 1[,]3,∞ [

c (-1) absolute minimum value , (3) absolute maximum value

3 a $x - 6y + 18 - \frac{\pi}{4} = 0$

b 2 minute

4 a e^{-2} b 30 cm , 60 cm

5 a $\frac{256}{15}\pi$ cubic units

b a = - 3 , b = 0

Test 6

1

1	2	3	4	5	6
b	a	a	b	a	b

2 a $-\frac{7}{20} \ln |2 - 5x^4| + c$

, $-\frac{3}{5} e^{-5x} + \pi \ln |x| + c$

3 a $= -\frac{1}{3}\csc^3 x + c$ b $(\frac{1}{2}, \frac{1}{\sqrt{2}})$

4 a absolute maximum value = 0

absolute minimum value = - 5

b b = -12

5 a $\frac{3}{20}$

b $\frac{5}{2}$ square units , 4π cubic units

Test 7

1

1	2	3	4	5	6
b	a	a	b	a	b

2 a $- x \cos x + \sin x + c$, $\frac{2}{3}(2\sqrt{2} - 1)$

b $y = x - \frac{\pi}{4}$

3 a The curve is convex downwards $x \in \mathbb{R}$

and no inflection points

b $6cm^3/sec$

4 a $\frac{116}{15}$ b $r = \frac{200}{\pi}$ meter

5 a **First:** absolute maximum value = 5

absolute minimum value = 1

Second: 4 square units

b 2π cubic units

Test 8

1 a $-\frac{3}{2}$ b $7 \sec x \tan x \, e^{\sec x}$

c (0, 0) d $\int_2^4 f(x) f x$

e $\frac{32}{3}$ square units

f $\frac{1}{2}$

2 a $\frac{1}{3} x^3 + \frac{9}{2} x^2 + 27 + c$

$- e^{-x} [x^2 + 2x + 2] + c$

b $12x - y + 2 - 3\pi = 0$

3 a $\frac{13}{3}$

4 a absolute maximum value = 6

absolute minimum value = 1

b $-\frac{3}{4}$ m/min

5 a $\frac{\pi}{3}$ b **First:** 13 square units

Second: 57π cubic units

Test 9

1

1	2	3	4	5	6
a	d	a	b	c	c

2 a $\frac{3}{2} \ln |x^2 - 1| + c$

$e^{3x} [3x^2 - 2x + \frac{2}{3}] + c$

b $18° \; 26'$

3 b $y = 4x + 5$

4 a $= (\frac{7}{100})^{rad}/min$

b f(- 1) = 8 local maximum value

, f(1) = 4 value local minimum

, (0, 6) inflection point

5 a $y = \frac{1}{3} x + 2$ b (3, 3)

c y = x d $\frac{45}{2}\pi$ cubic units

Test 10

1 a e

b $6 \csc^2 x (5 - 2\cot x)^2$

c 3 d - 16

e zero f $\frac{16}{15}$

2 a $-\frac{1}{3} \ln |\cos (3x + 1)| + c$

$\frac{1}{18} (3x - x^3)^5 + c$

b **First:** $2x - 3y - 7 = 0$ **Second:** $\frac{1}{9}$

3 a convex upwards on]0, 1[

convex downwards on]-∞,0[,]1 ,∞ [

(0, 1) inflection point

b b = - 48

4 a $\frac{32}{3}$ square units b $\frac{5}{2}$ cm/c

5 a **First:** $y = \frac{1}{12}(x + 1) + 9$

Second: $(\frac{1}{2}, \frac{3}{2})$ inflection point

b $\frac{23}{14}\pi$ cubic units

تم الطبع بالشروق الحديثة ـ القاهرة
بالمواصفات الفنية الآتية

عدد الصفحات بالغلاف : ١٧٢ صفحة

المقاس : ٥٧ × ٨٢ سم

نوع الورق : لا يقل الداخلى عن ٨٠ جرام والغلاف ٢٠٠ جرام

ألوان الطبع : ٤ لون للداخلى والغلاف

جميع حقوق الطبع محفوظة لوزارة التربية والتعليم
داخل جمهورية مصر العربية

الشروق
الحديثة للطباعة والتغليف
القاهرة : ٨ شارع سيبويه المصرى ـ ت : ٢٤٠٢٣٣٩٩ ـ فاكس : ٢٤٠٣٧٥٦٧ (٠٢)
مدينة العبور ـ المنطقة الصناعية